IN THE NAME OF ALLAH
THE MOST GRACIOUS
THE MOST MERCIFUL

Supersoluble groups of Wielandt length two

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To the last of prophets of ALLAH,

MUHAMMAD

(Peace and blessings of ALLAH be upon him and his family)

ıd his humble follower:

Syed Abul A'Ala Maududi.

Declaration

The work in this thesis is my own, except where otherwise stated.

Asif Ali

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Abstract

The Wielandt subgroup of a finite group is the set of its elements normalising every subnormal subgroup, and is known to be non-trivial in a non-trivial finite group. Iteration of this concept enables a Wielandt length to be defined for every finite group. The finite soluble groups of Wielandt length one are completely characterised.

This thesis is mainly directed at the classification of finite soluble groups of Wielandt length two. A characterisation of finite supersoluble groups of order coprime to six in this class is successfully given here. This generalises the characterisation by Ormerod of the nilpotent groups of odd order (Elizabeth A. Ormerod, *Groups of Wielandt length two,* Math. Proc. Camb. Phil. Soc. 110 (1991) 229-244).

The Wielandt structure of supersoluble groups, in general, is also investigated in this thesis. A relation between the Wielandt length of a supersoluble group and different invariants of its Sylow subgroups is established here. It is proved that if all Sylow subgroups of G have Wielandt length at most n, then G has Wielandt length at most $n + 1$ where $n = 1, 2$. It is also proved that if G is a supersoluble group and n is the maximum of the nilpotency classes of the Sylow subgroups of G, then G has Wielandt length at most $n + 1$, for all n.

Each subnormal subgroup of a finite soluble group G has Wielandt length at most the Wielandt length of G. This is not true in general for arbitrary subgroups of G. One of the results here concerns the finite soluble groups for which the groups, and all their factor groups, have the property that their subgroups have Wielandt length bounded by the Wielandt length of their parent group. It is proved that the groups of p-length one for all primes p have this property, and a result in the converse direction is also proved.

Contents

Notation

$\omega(G)$	Wielandt subgroup of G
$\omega^p(G)$	Local Wielandt subgroup of G
$\omega_i(G)$	i-th Wielandt subgroup of G
$\kappa(G)$	Norm of G
$\lvert G\rvert$	Number of elements of the group G
$\lvert\alpha\rvert$	Order of α
$\mathcal{W}_n$	Class of groups having Wielandt length at most n
$\alpha(x)$ or x^α	Image of x under α
S_n	Symmetric group on n letters
$[x, y]$	$x^{-1}y^{-1}xy$
$[H, K]$	$\langle [h, k] : h \in H, k \in K\rangle$
$[H, nK]$	$[H, \underbrace{K, \ldots, K}_{n}]$
$G' = [G, G]$	Derived subgroup of G
$G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$	Terms of the derived series of G
$N \lhd G$	N is a normal subgroup of G
$H \subseteq G$	H is a subgroup of G
$Z(G)$	Centre of G
$\gamma_i(G)$	Terms of the lower central series of G
$Z_i(G)$	Terms of the upper central series of G
$F(G)$	Fitting subgroup of G
$\Phi(G)$	Frattini subgroup of G
$\lvert G : H\rvert$	Index of the subgroup H in the group G
$C_G(A)$, $N_G(A)$	Centraliser, normaliser of A in G

$Hom(G,H)$	Set of homomorphisms from G to H
$End(G)$	Set of endomorphisms of G
$Aut(G)$	Automorphism group of G
$Paut(G)$	Power automorphism group of G
$O_\pi(G)$, $O_{\pi'}(G)$	Maximal normal π, π'-subgroup of G
$G^{\mathcal{T}}$	$\mathcal{T}$-residual of G
H^G	Normal closure of H in G
C_n	Cyclic group of order n
$A \times B$	Direct product of A and B
$\Omega_r(G)$	$\langle y \in G : y^{p^r} = 1 \rangle$
$\Gamma(G)$	$\langle y^p : y \in G \rangle$
$\sigma(G)$	Socle of G
$G(A,B,\theta)$	Semidirect product of A by B under the homomorphism θ
A_q	Sylow q-subgroup of A
S sn G	S is a subnormal subgroup of G
A wr B	Wreath product of A and B
$GL(V)$	Group of non-singular linear transformations of a vector space V
$GL_n(R)$, $SL_n(R)$	General linear and special linear groups
$A *_{N_k} B$	k-th nilpotent product of A and B

Chapter 1

Introduction

1.1 Preliminary comments

This thesis concerns one of the 'natural' characteristic subgroups of a group, the so called Wielandt subgroup. This is defined to be the subgroup consisting of those elements of a group G which normalise each subnormal subgroup of G. It is denoted by $\omega(G)$. An informal way of regarding the Wielandt subgroup is that it is yet another generalisation of the idea of centre of a group.

Of course a closer generalisation of centre, in this spirit, would be the subgroup of elements normalising every subgroup, the *norm* $\kappa(G)$ of Baer [3]. The advantage of Wielandt's subgroup over that of Baer, is that it is non-trivial in a much wider class of groups: Wielandt [17] showed that if N is a minimal normal subgroup of a group G which satisfies the minimal condition on its subnormal subgroups, then N is contained in $\omega(G)$. In particular, $\omega(G)$ is non-trivial in every finite G. The norm, and the centre, on the other hand, may be trivial in a finite group.

The Wielandt series of a group G is defined by setting $\omega_0(G) = 1$ and, for each $i \geq 1$, $\omega_i(G)/\omega_{i-1}(G) = \omega(G/\omega_{i-1}(G))$. For a finite group G, $\omega(G)$ is non-trivial and so $\omega_n(G) = G$ for some integer n. The smallest such n is called the Wielandt length of G. The factors in the Wielandt series are all T-groups, groups of Wielandt length one. Much is known about the structure of T-groups (Gaschutz [9], Zacher [18], Robinson [14] and Peng [13]). In particular finite soluble T-groups are completely characterised: Robinson [14]. For the purpose of this introduction, it is sufficient to note that they are metabelian of a very restricted type. The fact that a finite soluble group has an ascending series whose factors are so well understood often enables global, structural information to be inferred. Such results are to be found in, for example, Camina [5], Bryce and Cossey [4] for non-nilpotent groups, and in Ormerod [11], [12] for nilpotent groups.

This result of Ormerod [12] characterises the nilpotent groups of odd order and Wielandt length two. The present thesis is, in part, an attempt to extend this to finite soluble groups of Wielandt length two. The groups we deal with throughout are usually finite and soluble.

Note that the results in chapters 4 and 5 have also appeared in Research Reports [1] and [2].

1.2 Summary of results

In this section we outline the main results obtained in the thesis with precise references to their place of occurrence later.

The following result is basic to much of what follows.

Theorem 4.1.5 *Let $G = BA$ where A is a normal nilpotent subgroup of G such that $(|A|, |B|) = 1$.*

If P is the set of those elements $x \in \omega(B)$ such that $a \mapsto a^x$ is a power automorphism of A, then

$$\omega(G) = P\omega(A).$$

We observe, for example, that if G is a supersoluble group, and p is the largest prime dividing $|G|$, then a Sylow p-subgroup A is normal, and if B is a Hall p'-subgroup of G, then $G = BA$. Theorem 4.1.5 therefore has immediate application to supersoluble groups.

For soluble groups in general, several results in the literature use functions of Wielandt length to bound invariants such as derived length, Fitting length and nilpotency class (for example see Camina [5], Bryce and Cossey [4] and Ormerod [11]).

In chapter four we look at an inverse problem: can we bound Wielandt length as a function of numerical invariants of Sylow subgroups? The invariants for the Sylow subgroups we consider are Wielandt length, and nilpotency class. The results we obtain are the following:

Theorem 4.2.2 *Let G be a supersoluble group. If all Sylow subgroups of G have Wielandt length at most n, then G has Wielandt length at most $n+1$, where $n = 1, 2$.*

Theorem 4.2.3 *If G is a supersoluble group and n is the maximum of the nilpotency classes of the Sylow subgroups of G, then G has Wielandt length at most $n+1$.*

An example is given in section 3 of chapter 4 which shows that the bound given in Theorem 4.2.3 is best possible. We give another example to prove that this bound on the Wielandt length of supersoluble groups is not valid for soluble groups in general.

We know that homomorphic images and subnormal subgroups of a soluble group G have Wielandt length at most the Wielandt length of G, but in general the Wielandt length of a subgroup of G is not bounded by the Wielandt length of G. An example in section 6 of Cossey [7] gives a soluble group having Wielandt length two with a subgroup of arbitrarily large Wielandt length. On the other hand, in some classes of groups, the Wielandt length of subgroups *is* bounded by the Wielandt length of the group: for example in the class of T-groups and in the class of nilpotent groups.

Chapter 5 investigates the class $\mathcal{K}$ of soluble groups G such that each subgroup H of G has Wielandt length at most the Wielandt length of G. We conjecture that a group G belongs to $\mathcal{K}$ if and only if all Sylow subgroups of G have Wielandt length at most that of G. We denote by $\mathcal{V}$ the class of groups G whose Sylow subgroups have Wielandt length bounded by that of G. In other words we conjecture that $\mathcal{V} = \mathcal{K}$. It is easy to see that $\mathcal{K} \subseteq \mathcal{V}$ but the proof of the converse seems difficult.

An even stronger condition on a group is the following:

Suppose that, for each subgroup H of a group G and normal subgroup N of G,

$$(HN/N) \cap \omega(G/N) \subseteq \omega(HN/N).$$

Denote by $\mathcal{L}$ the class of all groups with this property. Note that the classes of T-groups and nilpotent groups have this property. It is also easy to see that

$$\mathcal{L} \subseteq \mathcal{K}.$$

However not all $G \in \mathcal{K}$ belong to $\mathcal{L}$: for example S_4, the symmetric group on four letters, is in $\mathcal{K}$ but does not belong to $\mathcal{L}$. We are able to characterise $\mathcal{L}$ as follows:

Theorem 5.2.5 *$\mathcal{L}$ is the class of groups of p-length one for all primes p.*

This immediately means that the class of supersoluble groups is contained in $\mathcal{K}$, since each supersoluble group G has p-length one for all primes p.

In chapter six we give our characterisation of supersoluble groups having Wielandt length two: Theorems 6.3.5 and 6.3.6. To state it here would require the introduction of too much notation at this stage. A fundamental feature of the proof is the following result.

Theorem 6.1.7 *Let N be the nilpotent residual of a supersoluble group G of odd order and Wielandt length two. Then N is complemented in G.*

Chapter 2

Definitions, notation and basic facts

In this chapter we give definitions, notation and basic facts which are used throughout the thesis. This is partly for fixing terminology, partly to provide an easy reference to results we need to quote from the literature, and partly for setting up notation. As a general rule we follow the notation and terminology used by Doerk and Hawkes in [8].

Other references will be to Robinson [14] and Huppert [10].

It is important to note that almost all groups which we will consider in this thesis are finite: some standard results we quote, and methods we use, involve infinite groups but these will be applied in this thesis only in a finite context.

The reader should be able to bypass much of section 1 at a first reading, using it as a reference for results quoted in later chapters.

2.1 Basic definitions and notation

Derived subgroup For subgroups H, K in a group G we write

$$[H, K] = \langle [h, k] : h \in H, k \in K \rangle.$$

It is sometimes convenient to write $[H, nK] = [H, \underbrace{K, \ldots, K}_{n}]$, with a left norming convention:

$$[H, nK] = [H, (n-1)K], K].$$

The subgroup of a group G, generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ for $x, y \in G$ is called the *derived subgroup* of G. It is denoted by G'. The derived subgroup is fully-invariant and it is easy to see that $G' = [G, G]$.

By repeatedly forming derived subgroups a descending series of fully-invariant subgroups is generated:

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \dots$$

where $G^{(i)} = (G^{(i-1)})'$ for $i \geq 1$. This is called the *derived series.*

Soluble Groups A group G is said to be *soluble* if, for some positive integer n, $G^{(n)} = 1$.

It is well known that a finite group G is soluble if and only if it has a chief series whose factors are abelian. By the Jordan-Hölder Theorem (3.1.4 of Robinson [14]), it follows immediately that a finite group G is soluble if and only if every chief factor of G is abelian.

Derived length If G is a soluble group, then the minimum length of a derived series in G is called the *derived length* of G. Thus the groups having derived length one are just the abelian groups. A soluble group with derived length at most two is said to be *metabelian.*

Supersoluble Groups A group G is said to be *supersoluble* if there exists a normal series, that is a series of subgroups normal in G,

$$1 = G_0 \lhd G_1 \lhd \dots \lhd G_n = G$$

such that each factor G_i/G_{i-1} is cyclic. Equivalently, relying again on the Jordan-Hölder Theorem, we can say that a finite group G is supersoluble if and only if each chief factor of G is cyclic. For example all finitely generated nilpotent groups are supersoluble. Supersoluble groups are, of course, soluble.

Nilpotency Class Let G be a group. The *lower central series* of G is defined as

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots$$

where $\gamma_i(G) = [G, \gamma_{i-1}(G)]$ for $i = 2, 3, \ldots$. All the terms in the lower central series are characteristic subgroups of G. If for some k we have $\gamma_{k+1}(G) = 1$ then G is said to be *nilpotent* and if k is the smallest such integer for which this happens, then k is called the *nilpotency class* of G.

The *upper central series* for G is defined by

$$1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \ldots$$

where $Z_i(G)$ is the characteristic subgroup of G defined by:

$$Z_1(G) = Z(G),$$

the centre of G, and

$$Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1})$$

for $i = 1, 2, \ldots$. It is well known that G is nilpotent of nilpotency class k if and only if $Z_k(G) = G$ but $Z_{k-1}(G) \neq G$.

Fitting Subgroup The subgroup generated by all the normal nilpotent subgroups of a finite group G is called the *Fitting subgroup* of G. It is denoted by $F(G)$. $F(G)$ is the unique largest normal nilpotent subgroup of G. If G is a non-trivial soluble group, $F(G)$ contains every minimal normal subgroup of G and hence is not trivial.

Theorem 2.1.1 [5.2.9, Robinson [14]] *Let G be a finite soluble group. Then $F(G)$ is the intersection of the centralizers of the chief factors of G.* □

Frattini Subgroup The *Frattini subgroup* of a finite group G is defined to be the intersection of all the maximal subgroups of G. It is obviously characteristic. It is denoted by $\Phi(G)$. An element g is called a *non-generator* if $G = \langle g, X \rangle$ always implies that $G = \langle X \rangle$, where X is a subset of G.

Theorem 2.1.2 [5.2.12, Robinson [14]] *In any group G, the Frattini subgroup is equal to the set of non-generators of G.* □

Theorem 2.1.3 [Theorem A.9.14, Doerk and Hawkes [8]] *Let G be a finite group generated by d elements. Then the following statements are true.*

1) $|Aut(G)|$ divides $|Aut(G/\Phi(G))||\Phi(G)|^d$.

2) If $\alpha \in Aut(G)$ and $[G, \alpha] \subseteq \Phi(G)$, then the order of α divides $|\Phi(G)|^d$; in particular, if $(|\alpha|, |\Phi(G)|) = 1$, then $\alpha = 1$. □

Here $[G, \alpha]$ is defined to be the subgroup $\langle g^{-1}g^{\alpha} : g \in G\rangle$.

π-Groups Let π be a set of primes. A *π-number* is a positive integer whose prime divisors belong to π. An element of a group G is called a *π-element* if its order is a π-number. A group G is said to be π-group if every element of G is a π-element. The notation π' is used for the set of all primes not in π. In particular, if p is a prime then p' denotes the primes other than p.

If G is a finite group, a π-subgroup H of G is called *Hall π-subgroup* of G if $|G : H|$ is a π'-number.

p-Length Let p be a prime. The subgroup generated by all the normal π-subgroups of a group G is denoted by $O_\pi(G)$. It is the unique maximal normal π-subgroup of G.

The *upper p-series* is generated by repeatedly applying $O_{p'}$ and O_p. This is, then, the series

$$1 = P_0 \lhd N_0 \lhd P_1 \lhd N_1 \lhd \ldots \lhd P_m \lhd N_m \lhd \ldots$$

defined by $P_0 = 1$ and, for $i \geq 0$, $N_i/P_i = O_{p'}(G/P_i)$ and $P_{i+1}/N_i = O_p(G/N_i)$. In a finite soluble group G, this series ascends properly at every step until either $N_r = G$ or $P_r = G$, for some r. The number of non-trivial p-factors P_i/N_{i-1} in this series is called the *p-length* of G.

Formations A class of finite groups $\mathcal{T}$ is said to be a *formation* if every homomorphic image of a $\mathcal{T}$-group is a $\mathcal{T}$-group, and if $G/(M \cap N)$ belongs to $\mathcal{T}$ whenever G/M and G/N belong to $\mathcal{T}$. The classes of soluble groups, nilpotent groups and supersoluble groups are formations.

A formation $\mathcal{T}$ is said to be *saturated* if $G \in \mathcal{T}$ whenever $G/\Phi(G) \in \mathcal{T}$. The classes of soluble groups, nilpotent groups and supersoluble groups are saturated formations.

If G is a finite group, and $\mathcal{T}$ a formation, the $\mathcal{T}$-residual in G is the intersection of all normal subgroups N of G for which $G/N \in \mathcal{T}$. The $\mathcal{T}$-residual is the smallest normal subgroup of G whose factor group is in $\mathcal{T}$. We denote the $\mathcal{T}$-residual of G by $G^{\mathcal{T}}$.

Theorem 2.1.4 [Theorem 4.5.18, Doerk and Hawkes [8]] *Let $\mathcal{T}$ be a saturated formation and A be the $\mathcal{T}$-residual of a finite group G. If A is abelian, then A is complemented in G.* □

Regular p-groups A p-group G is *regular* if for all $x, y \in G$, we have

$$x^p y^p = (xy)^p \prod c_i^p$$

for some suitable $c_i \in \langle x, y \rangle'$.

Theorem 2.1.5 [Satz 3.10.2 (a), Huppert [10]] *Let G be a p-group. If the nilpotency class of G is less than p, then G is regular.* □

Theorem 2.1.6 [Satz 3.10.6 (b), Huppert [10]] *Let G be a regular p-group. Then $[x^{p^k}, y^{p^n}] = 1$ if and only if $[x, y]^{p^{k+n}} = 1$.* □

Quotations

Now we list some results which we need in proving main results in coming chapters.

Theorem 2.1.7 [A.1.3, Doerk and Hawkes [8]] *Let H, K, L be subgroups of a group and assume that $K \subseteq L$. Then $(HK) \cap L = (H \cap L)K$.* □

Theorem 2.1.8 [Razmyslov's Theorem, (ch.4 of Vaughan Lee [16])] *If n is a prime power and $n \geq 4$, then there exists an insoluble locally finite group of exponent n.* □

Theorem 2.1.9 [8.1.4, Robinson [14]] *Let M and N be simple modules over a ring R. If M and N are not isomorphic, $Hom_R(M, N) = 0$. Also*

$$Hom_R(M, M) \cong End_R(M)$$

is a division ring. □

Theorem 2.1.10 [Proposition A.12.5, Doerk and Hawkes [8]] *If Q is a π'-group of operators for a π-group P, then*

1) $P = [P, Q]C_P(Q)$.

2) $[P, Q] = [[P, Q], nQ]$ *for all* $n \geq 1$.

3) *if P is abelian, then*

$$P = [P, Q] \times C_P(Q).$$

□

Theorem 2.1.11 [Theorem A.11.6, Doerk and Hawkes [8]] *Let A be a finite abelian p-group, and let B be a group of operators for A with $(|B/C_B(A)|, |A|) = 1$. Then A has a direct composition*

$$A = A_1 \times \ldots \times A_s$$

into B-admissible subgroups A_i with the following properties for each $i = 1, \ldots, s$:

1) *A_i is indecomposable as a B-module;*

2) *$A_i/\Phi(A_i)$ is an irreducible B-module;*

3) *A_i is homocyclic.* □

Theorem 2.1.12 [Lemma A.9.10, Doerk and Hawkes [8]] *Let H/K be an abelian chief factor of a finite group G. Then*

1) *H/K is either complemented or*

$$H/K \subseteq \Phi(G/K),$$

and

2) *if H/K is complemented, then each complement is a maximal subgroup of G.* □

Theorem 2.1.13 [Theorem A.10.6 (c), Doerk and Hawkes [8]] *Let G be a finite group. Then $F(G/\Phi(G)) = F(G)/\Phi(G)$ and it is the product of the abelian minimal normal subgroups of $G/\Phi(G)$, all of which are complemented.* □

Theorem 2.1.14 [Satz 3.9.4, Huppert [10]] *Let G be the free group on the free generators x and y. Then there exist elements $c_i \in \gamma_i(G)$ with $i = 2, 3, \ldots$ such that, for all $m > 0$, we have*

$$x^m y^m = (xy)^m c_2^{\binom{m}{2}} \ldots c_{m-1}^{\binom{m}{m-1}} c_m.$$

□

Theorem 2.1.15 [Theorem A.9.6 (a), Doerk and Hawkes [8]] *Let G be a finite p-group. Then $\Phi(G) = G'\Gamma(G)$, where $\Gamma(G) = \langle g^p | g \in G \rangle$.* □

Theorem 2.1.16 [Schenkman [15]] *In an arbitrary group G, we have*

$$\kappa(G) \subseteq Z_2(G).$$

□

This result of Schenkman will usually be applied when G is a finite nilpotent group. In this case it says that

$$\omega(G) \subseteq Z_2(G).$$

2.2 Subnormal structure

As is clear from the title of the thesis, most of the time we will be discussing the Wielandt structure or the subnormal structure of supersoluble groups. We now give some notes on some frequently used definitions and related facts about subnormal structure.

Subnormal Subgroup A subgroup H of a group G is called *subnormal* in G if there exists a chain of subgroups $H_0, H_1, \ldots, H_r$ of G such that

$$H = H_0 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_{r-1} \lhd H_r = G.$$

This is called a *subnormal chain* from H to G. If H is subnormal in G, we shall write H sn G. The length of the shortest subnormal chain from H to G is called the *defect* of H in G.

The Wielandt Subgroup The subgroup consisting of those elements of a group G which normalise each subnormal subgroup of G is called the *Wielandt subgroup* of G. It is denoted by $\omega(G)$.

This subgroup is named after Wielandt who defined it in 1958, in [17]. In the same paper he showed that for any group G and any minimal normal subgroup N of G satisfying the minimal condition on subnormal subgroups, $\omega(G)$ contains N. Thus for a finite group G, $\omega(G)$ contains the socle of G, the subgroup generated by all the minimal normal subgroups of G. We denote the socle of G by $\sigma(G)$. In a finite group G, $\sigma(G)$ and hence $\omega(G)$, is non-trivial. We define the *ascending Wielandt series* for G as follows. Write $\omega_0(G) = 1$, and for $i \geq 1$, define the subgroup $\omega_i(G)$ of G inductively by: $\omega_i(G)/\omega_{i-1}(G) = \omega(G/\omega_{i-1}(G))$. By the remark above there is, in a finite group G, a smallest n such that $\omega_n(G) = G$. We call n the *Wielandt length* of G. The class of groups having Wielandt length at most n is denoted by $\mathcal{W}_n$.

The groups for which the Wielandt length is one are called *T-groups* which are of special interest.

T-Groups A group G is called a *T-group* if each subnormal subgroup of G is normal. In other words G is a T-group if and only if $\omega(G) = G$. This means that a group G is a T-group if and only if it has Wielandt length exactly one. For example the quaternion group of order 8 is a T-group and the smallest finite group that is not a T-group is the dihedral group of order 8. The Wielandt subgroup of a finite group G is always a T-group.

Dedekind Groups A group G is called *Dedekindian* or *Dedekind* if each subgroup of G is normal in G. A nilpotent group is a T-group if and only if it is Dedekindian. A non-abelian Dedekindian group is called *Hamiltonian*. The following result characterises all Dedekind groups.

Theorem 2.2.1 [5.3.7, Robinson [14]] *All the subgroups of a group G are normal if and only if G is abelian or the direct product of a quarternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order.* □

We mentioned in the introduction that a detailed picture of the structure of T-groups is given in 13.4.6 of Robinson [14]. Before we state that result, we define power automorphisms and give some results regarding them.

Power Automorphisms An automorphism of a group G is said to be *power automorphism* if it maps each element x to x^n for some positive integer $n = n(x)$. The group of power automorphisms of G is denoted by $Paut(G)$. It is central in $Aut(G)$, the group of all automorphisms of G. For a cyclic group G, of course, we have $Aut(G) = Paut(G)$.

A power automorphism is called *universal* if it maps all elements to the same power, say n. A universal power automorphism is therefore completely determined by this integer n.

Theorem 2.2.2 [Theorem 2.2.1, Cooper [6]] *Every power automorphism of a group is central.* □

Theorem 2.2.3 [Theorem 5.3.1, Cooper [6]] *If G is a locally finite group whose Sylow subgroups are regular, every power automorphism of G is locally universal.* □

Now we give the classification theorem for T-groups alluded to above. We use this result often.

Theorem 2.2.4 [13.4.4,13.4.6, Robinson [14]]

a) Let A, B be, respectively, abelian of odd order, and Dedekindian, and suppose their orders are coprime. Also let

$$\theta : B \longrightarrow Paut(A)$$

be a homomorphism with the property that for each prime p dividing $|A|$, there is an element b_p of B such that b_p^θ acts non-trivially on the p-component of A.

Then the semi-direct product

$$G(A, B, \theta)$$

of A by B under θ is a T-group.

b) Conversely every finite soluble T-group is isomorphic to one of the groups $G(A, B, \theta)$. □

It is also important to note the following fact about T-groups.

Theorem 2.2.5 [13.4.7, Robinson [14]] *A subgroup of a finite soluble T-group is a T-group.* □

We have found it necessary to use a local version of T-groups. This idea was developed and used by Bryce and Cossey in [3]. For each prime p we consider the groups for which each p'-perfect subnormal subgroup is normal. Here a subgroup is p'-perfect if it has no non-trivial factor groups of order coprime to p. The results Bryce and Cossey obtained in [3] are local analogues for those for T-groups; the corresponding results for T-groups come as corollaries of their results.

A number of results are more transparent if framed in terms of the *local Wielandt subgroup*, defined for each prime p as the set of elements in G normalising every p'-perfect subnormal subgroup of G. We denote by $\omega^p(G)$ the local Wielandt subgroup in a finite group G for the prime p. Here are some results from Bryce and Cossey [4] which we use mostly in chapter four and five. In these results G is always finite and soluble.

Theorem 2.2.6 [Lemma 3.1, Bryce and Cossey [4]] *If N is a normal subgroup of a group G, then for all primes p,*

$$\omega^p(G)N/N \subseteq \omega^p(G/N), \omega(G)N/N \subseteq \omega(G/N)$$

and

$$\omega^p(G) \cap N \subseteq \omega^p(N), \omega(G) \cap N \subseteq \omega(N).$$

□

Theorem 2.2.7 [Lemma 3.2, Bryce and Cossey [4]] *Let N be a normal subgroup of a group G.*

1) If N is a p'-group, then $N \subseteq \omega^p(G)$ and

$$\omega^p(G)/N = \omega^p(G/N).$$

2) If G/N is a p'-group then

$$\omega^p(N) = \omega^p(G) \cap N.$$

In particular $O_{p'}(G) \subseteq \omega^p(G)$ and

$$\sigma(G/O_{p'}(G)) \subseteq \omega^p(G)/O_{p'}(G).$$

3) If G/N is a p'-group, then

$$O_p(\omega^p(N)) = O_p(\omega^p(G)).$$

□

Theorem 2.2.8 [Theorem 3.7, Bryce and Cossey [4]] *If G is any group, then*

$$\omega(G) = \cap_p \omega^p(G),$$

the intersection being taken over all primes p. □

Theorem 2.2.9 [Theorem 3.8, Bryce and Cossey [4]] *1) If G/N is a p'-group, then*

$$O_p(\omega^p(N)) = O_p(\omega(G)).$$

2) For a prime p we have

$$\omega(G/O_{p'}(G)) = \omega^p(G)/O_{p'}(G).$$

□

Theorem 2.2.10 [Lemma 3.10, Bryce and Cossey [4]] *If G is any group, then*

$$\omega(G)/F(\omega(G)) \subseteq Z(G/F(\omega(G))).$$

□

2.3 Supersoluble groups

The aim of this section is to record some results concerning supersoluble groups which will be needed in the sequel. Theorem 2.3.6 appears to be new.

Theorem 2.3.1 [5.4.10, Robinson [14]] *If G is a supersoluble group, $F(G)$ is nilpotent and $G/F(G)$ is a finite abelian group. In particular G' is nilpotent.* □

Theorem 2.3.2 [5.4.8, Robinson [14]] *If G is a finite supersoluble group, there is a normal series*

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

in which each factor is cyclic of prime order and with non-increasing orders $|G_i/G_{i-1}|$ from the left. □

The following are the immediate consequences of Theorem 2.3.2.

Corollary 2.3.3 *If p is the largest prime dividing the order of a finite supersoluble group G, then a Sylow p-subgroup of G is normal.* □

Corollary 2.3.4 *If A is a normal Sylow subgroup of a supersoluble group G, then*

1) A is complemented in G.

2) If B is a complement for A in G, then $B/C_B(A)$ is abelian.

Proof 1) Since A is a Sylow subgroup of G, G/A has order coprime to that of A. Hence by the Schur-Zassenhaus Theorem, A is complemented in G.

2) Now $B' \subseteq G' \subseteq F(G)$ (using Theorem 2.3.1). But as B is coprime in order to A, and B' and A lie in $F(G)$, we have $[B', A] = 1$ and hence $B' \subseteq C_B(A)$. Thus $B/C_B(A)$ is abelian. □

The next theorem gives necessary and sufficient conditions for a semidirect product of a nilpotent group by a supersoluble group to be supersoluble. But first we prove the following lemma which we use in the proof.

Lemma 2.3.5 *Let H be a supersoluble group in which, for some prime q dividing $|H|$, a Hall q'-subgroup of H has exponent dividing $q-1$. Then a Sylow q-subgroup of H is normal in H.*

Proof We use induction on $|H|$. If $|H| = q$, there is nothing to prove. So suppose that $|H| > q$ and that the result is true in every group satisfying the hypothesis and of order less than $|H|$.

Let N be a minimal normal subgroup of H. Since H is supersoluble, N is cyclic of prime order. First consider the case that $|N| = q$. If q^2 does not divide $|H|$, then N is a Sylow q-subgroup and

$$N = O_q(H).$$

So suppose that q^2 divides $|H|$. The group H/N satisfies the hypothesis of the theorem, so $O_q(H/N)$ is a normal Sylow q-subgroup of H/N, by induction. We may write $K/N = O_q(H/N)$ where K is a normal subgroup of H. Note that $|K| = |N||O_q(H/N)|$, so K is a Sylow subgroup of H. Also $K = O_q(H)$.

It remains to consider the case that $|N| \neq q$. Then q divides $|H/N|$ and H/N does satisfy the hypothesis of the theorem. By induction, therefore, $O_q(H/N)$ is the unique Sylow q-subgroup of H/N. Again write $K/N = O_q(H/N)$ so that $K \lhd H$.

Let L be a Sylow q-subgroup of K, so that $K = LN$. Note also that L is a Sylow q-subgroup of H. By hypothesis, $|N|$ divides $q - 1$, so it follows that $[L, N] = 1$. Therefore L is a characteristic subgroup of K which is then normal in H. That is, $L = O_q(H)$ is a Sylow q-subgroup of H.

This completes the induction, and the proof of the lemma. □

Theorem 2.3.6 *Let $G = BA$ where A is normal in G and $A \cap B = 1$ with A nilpotent and B supersoluble. Then G is supersoluble if and only if for every prime q dividing $|A|$, $B_{q'}/C_{B_{q'}}(A_q)$ is abelian of exponent dividing $q - 1$, where $B_{q'}$ is a Hall q'-subgroup of B.*

Proof Let A_q be a Sylow q-subgroup of A. Since A_q is characteristic in A, it is normal in G. Hence $\Phi(A_q)$ is similarly normal in G. We therefore have a series

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_k = A_q, \tag{2.1}$$

part of a chief series of G. We define, for each $i \in \{1, 2, \ldots, k\}$, a homomorphism $\psi_i : B \longrightarrow Aut(G_i/G_{i-1})$ as follows.

For each $b \in B$ we may check that the function $\chi_b : gG_{i-1} \rightarrow g^b G_{i-1}$ is well defined, and that it is an automorphism of G_i/G_{i-1}. Then we may check that

$$\psi_i(b) = \chi_b,$$

for $b \in B$ defines a homomorphism, as claimed. Let us denote

$$ker\psi_i = C_i$$

for $1 \leq i \leq k$. We also write $C = \cap_{i=1}^{k} C_i$. Note that $C_{B_{q'}}(A_q) \subseteq C_i$ for all $1 \leq i \leq k$. So $C_{B_{q'}}(A_q) \subseteq C$. Also, by Theorem 2.1.10 (2),

$$[A_q, C \cap B_{q'}] = 1.$$

Therefore $C \cap B_{q'} = C_{B_{q'}}(A_q)$.

Choose an arbitrary $i \in \{1, 2, \ldots, k\}$. We observe that $V = G_i/G_{i-1}$ may be regarded as a vector space over the field $GF(q)$ of q elements, say of dimension d. It follows that

$$Aut(G_i/G_{i-1}) \cong GL(V) \cong GL_d(q),$$

the general linear group of degree d over $GF(q)$. Hence $B_{q'}/C_i \cap B_{q'} \cong \psi_i(B_{q'})$ is isomorphic to a subgroup of $GL_d(q)$.

Now suppose that G is supersoluble and so each G_i/G_{i-1} is cyclic whence $d = 1$. So $GL_d(q)$ is isomorphic to the multiplicative group of $GF(q)$. In this case, therefore, $B_{q'}/C_i \cap B_{q'}$ is isomorphic to a subgroup of the multiplicative group of $GF(q)$ which is cyclic of order $q-1$. Hence $B_{q'}/C_i \cap B_{q'}$ is cyclic of order dividing $q-1$.

Since $B_{q'} \cap C = \cap_{i=1}^{k} C_i \cap B_{q'}$, it follows that $B_{q'}C/C \cong B_{q'}/B_{q'} \cap C$ is isomorphic to a subdirect product of the groups $B_{q'}/C_i \cap B_{q'}$. Therefore $B_{q'}/C_{B_{q'}}(A_q) = B_{q'}/C \cap B_{q'}$ is abelian of exponent dividing $q-1$, as required.

Conversely, suppose that $B_{q'}/C_{B_{q'}}(A_q)$ is abelian of exponent dividing $q-1$. We claim that $d = 1$. To this end consider the group B/C_i. This is isomorphic to a subgroup X of $GL_d(q)$. Note that X is irreducible since G_i/G_{i-1} is a chief factor of G. However this means that $O_q(X) = 1$ (by B.3.12 of Doerk and Hawkes [8]).

On the other hand B/C_i is supersoluble. Now $B_{q'}C_i/C_i \cong B_{q'}/C_i \cap B_{q'}$ is a Hall q'-subgroup of B/C_i, and has exponent dividing $q-1$. It follows from Lemma 2.3.5 that a Sylow q-subgroup of B/C_i is normal in B/C_i. But $1 = O_q(X) \cong O_q(B/C_i)$, therefore q does not divide $|B/C_i|$, so $X \cong B/C_i = B_{q'}C_i/C_i$ is abelian of exponent dividing $q-1$.

Now we may regard $V = G_i/G_{i-1}$ as a module for X over $GF(q)$ and as such it is simple since G_i/G_{i-1} is a chief factor of G. Let $F = End_X(V)$. By Theorem 2.1.9, F is a finite division ring. Using Wedderburn's Theorem (B.3.17 of Doerk and Hawkes [8]), F is a field and therefore $F \cong GF(q^r)$ for some $r > 0$.

Since X is abelian, $X \subseteq F$. Therefore V is simple as an F-module. It follows that $F \cong V$ as $GF(q)X$-module and so $r = d$. However all elements $x \in X$ satisfy $x^{q-1} = 1$, and the only elements of F with this property lie in $GF(q)$. But F as $GF(q)$-module, is a direct sum of d one dimensional submodules and therefore V, as X-module, could be irreducible only if $d = 1$.

This shows that if $B_{q'}/B_{q'} \cap C$ is abelian of exponent dividing $q-1$, then $d = 1$. Hence G_i/G_{i-1} is cyclic for all $1 \leq i \leq k$. Since this holds for all primes q dividing $|A|$ we can extend (2.1) to a chief series $1 = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_t = A$ of A whose factors are all cyclic. But since B is supersoluble, B has a chief series $1 = B_0 \lhd B_1 \lhd B_2 \lhd \ldots \lhd B_j = B$ such that B_i/B_{i-1} $(1 \leq i \leq j)$ are all cyclic. Now

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_t = A \lhd B_1A \lhd B_2A \lhd \ldots \lhd BA = G$$

is a chief series of G whose factors are all cyclic. Hence G is supersoluble. □

Chapter 3

Groups of Wielandt length two: p-groups

In her article [12] Ormerod gave a characterisation of p-groups of Wielandt length two in the case that p is an odd prime. This result, which we state below, can be seen as motivated by Dedekind's characterisation of groups in which every subgroup is normal: see Theorem 2.2.1. The idea was to describe the $\mathcal{W}_2$-groups in terms of groups obviously in $\mathcal{W}_2$, namely groups of nilpotency class two, and a small number of 'sporadic' exceptions.

For the application we make in chapter six it will be necessary to fill in some detail in Ormerod's theorem [12] and for this purpose we have found it useful to complete some of the detail she omitted in [12]. Our account will be directed to the application we make in chapter six, where the groups will have order coprime to six: the 3-groups in Ormerod's work will therefore be omitted here.

This chapter is therefore an account of p-groups of Wielandt length two, where p is a prime greater than three, based on Ormerod and adding detail to it.

3.1 A $\mathcal{W}_2$-group of nilpotency class three

We begin the construction of one of Ormerod's 'sporadic' groups. Let $p > 3$ be a prime, and r a positive integer. Consider the cyclic groups $A = \langle a | a^{p^r} = 1\rangle$ and $X = \langle x | x^{p^{2r}} = 1\rangle$ of orders p^r and p^{2r} respectively. Define an automorphism of X as follows: $\theta : x \mapsto x^{1-p^r}$. As an element of the automorphism group of X, this has order p^r. This is because $(1-p^r)^{p^r} \equiv 1 \bmod p^{2r}$. So $|\theta|$ divides p^r; and on the other hand, $(1-p^r)^{p^s} \equiv 1 - p^{r+s} \not\equiv 1 \mod p^{2r}$ for $s < r$. Hence there is a homomorphism $\alpha : A \longrightarrow Aut(X)$ defined by $a^\alpha : x \mapsto x^{1-p^r}$. Form the semi-direct product H_0 of X by A under α. We see that H_0 has the following presentation:

$$H_0 = \langle a, x : a^{p^r} = x^{p^{2r}} = 1, [a, x] = x^{p^r} \rangle.$$

Since $[a, x, a] = [x^{p^r}, a] = [x, a]^{p^r} = 1$, we see that H_0 has nilpotency class two, so is regular by Theorem 2.1.5. Also $|H_0| = p^{3r}$.

Next let $Y = \langle y | y^{p^r} = 1 \rangle$ and $B = \langle b | b^{p^r} = 1 \rangle$ be cyclic groups of order p^r. Note that $H_0 \times B$ has the presentation

$$\langle a, x, b : a^{p^r} = x^{p^{2r}} = b^{p^r} = 1, [a, x] = x^{p^r}, [a, b] = [x, b] = 1 \rangle.$$

We aim to define a homomorphism $\beta : Y \longrightarrow H_0 \times B$ with the following properties:

$$y^\beta : x \mapsto xa, a \mapsto ab, b \mapsto b.$$

To ensure the existence of β we need only show that xa, ab and b satisfy the defining relations displayed for $H_0 \times B$, and use Van Dyck's Theorem:

$$(xa)^{p^r} = x^{p^r} a^{p^r} [a, x]^{p^r(p^r-1)/2} = x^{p^r}$$

and so

$$(xa)^{p^{2r}} = 1;$$

$$(ab)^{p^r} = a^{p^r} b^{p^r} = 1;$$

$$b^{p^r} = 1;$$

$$[ab, xa] = [ab, a][ab, x]^a = [b, a][a, x] = x^{p^r} = (xa)^{p^r};$$

$$[ab, b] = 1; [xa, b] = 1.$$

Write H_1 for the semi-direct product of H_0 by Y under β. Then H_1 has a presentation

$$H_1 = \langle a, b, x, y : a^{p^r} = b^{p^r} = x^{p^{2r}} = y^{p^{2r}} = 1,$$

$$[a, x] = x^{p^r}, [x, y] = a, [a, y] = b, [a, b] = [x, b] = [y, b] = 1 \rangle.$$

Note that $[x, y, y] = [a, y] = b$, and b is central, so

$$[x, y^{p^r}] = [x, y]^{p^r}[x, y, y]^{p^r(p^r-1)/2} = a^{p^r}b^{p^r(p^r-1)/2} = 1.$$

Also $[a, y^{p^r}] = [a, y]^{p^r} = b^{p^r} = 1$.

It follows that y^{p^r} is central. Since it is of order p^r we may pass from H_1 to its factor group modulo $\langle y^{-p^r}b\rangle$. We then have a group H presented as

$$\langle a, b, x, y : a^{p^r} = b^{p^r} = x^{p^{2r}} = y^{p^{2r}} = 1,$$

$$[a, x] = x^{p^r}, [x, y] = a, [a, y] = b, [a, b] = [x, b] = [y, b] = 1, y^{p^r} = b\rangle.$$

Applying several Tietze transformations we see that H has the presentation

$$H = \langle x, y : [x, y, x] = x^{p^r}, [x, y, y] = y^{p^r}, x^{p^{2r}} = y^{p^{2r}} = [x, y]^{p^r} = 1\rangle. \quad (3.1)$$

Note that, by its construction, $|H| = p^{5r}$. This group H is the one Ormerod [12] denotes $H(p, r)$. This is because H and $H(p, r)$ have the same order and Van Dyck's Theorem provides a homomorphism from $H(p, r)$ onto H.

Definition 3.1.1 *Let $H(p, r)$ denote the group H given by 3.1.*

The properties of $H(p, r)$ we need are summed up in the following theorem.

Theorem 3.1.2 *$H = H(p, r)$ is a group of order p^{5r} and nilpotency class three, with the following properties:*

1) H is regular;

2) $H/H' \cong C_{p^r} \times C_{p^r}$;

3) $\omega(H) = H' = Z_2(H)$, a subgroup of exponent p^r;

4) the exponent of $H/\gamma_3(H)$ is p^r.

Proof Note that (1) follows from Theorem 2.1.5, while H' has exponent p^r as a consequence of our construction of H. Since $C_{p^r} \times C_{p^r}$ has a pair of generators satisfying the relations 3.1, there is a homomorphism from H onto $C_{p^r} \times C_{p^r}$. On the other hand, from 3.1, we see that H/H' is a two generator abelian group of exponent dividing p^r, so the kernel can be no larger than H'.

To show that $\omega(H) = H'$ we first show that $H' \subseteq \omega(H)$. Each $h \in H$ may be written as $h = x^m y^n d$ where m, n are non-negative integers, and $d \in H'$. Then, using Theorem 2.1.14,

$$[x, y, h] = [x, y, x]^m [x, y, y]^n = x^{mp^r} y^{np^r} = (x^m y^n d)^{p^r} = h^{p^r}.$$

Therefore $[x, y] \in \omega(H)$. However $H' = \langle [x, y] \rangle \gamma_3(H)$, and $\gamma_3(H) \subseteq Z(H)$ which shows that $H' \subseteq \omega(H)$. To prove the converse suppose that there exists $h \in \omega(H)$ such that $h \notin H'$. We may suppose, without loss of generality, that $h = x^m y^n$ where p^r does not divide n. Also

$$[h, x] = [y^n, x] = [y, x]^n [y, x, y]^{n(n-1)/2} \in \langle x \rangle \cap H' = \langle x^{p^r} \rangle \subseteq \gamma_3(H).$$

But this requires $[y, x]^n \in \gamma_3(H)$ which is possible only if $p^r | n$, a contradiction. Hence $\omega(H) = H'$, as required.

To calculate $Z_2(H)$, we see that $H' \subseteq Z_2(H)$ since the class of H is three. If there is an element $h \in Z_2(H)$ such that $h \notin H'$, then we may assume that $h = x^m y^n$ where either p^r does not divide m or p^r does not divide n, say the latter. Then $1 = [y^n, x, x] = [y, x, x]^n$ whence $p^r | n$, a contradiction. Therefore (3) is proved.

To prove (4) note that the exponent of $H/\gamma_3(H)$ is, by (1), at least p^r. On the other hand it follows from the presentation (3.1) and Theorem 2.1.14, that all p^r-th powers are in $\gamma_3(H)$. Hence the exponent of $H/\gamma_3(H)$ is precisely p^r. □

We note in passing that, because of (3), $H(p, r) \in \mathcal{W}_2$. In fact $H(p, r)$ is one of the 'sporadic' groups of which we spoke in the introduction to this chapter: the other is a 3-group and is not relevant to us here.

3.2 Characterising p-groups in $\mathcal{W}_2$

Let us begin this section with a proof of the following result.

Theorem 3.2.1 *A nilpotent group of Wielandt length two has nilpotency class at most three.*

Proof Since a nilpotent group is the direct product of its Sylow subgroups, it is enough to show that a p-group $A \in \mathcal{W}_2$ has nilpotency class at most three.

When $p \geq 3$, this is easy: $A/\omega(A)$ is a nilpotent T-group and therefore abelian by Dedekind's Theorem (Theorem 2.2.1), so $A' \subseteq \omega(A)$. Also $\omega(A) \subseteq Z_2(A)$ by Theorem 2.1.16. Hence A has nilpotency class at most three.

When $p = 2$, the proof is not so direct. By Baer [3], if $\omega(A)$ is Hamiltonian, then $\omega(A) = A$ has nilpotency class two. Hence we may suppose that $\omega(A)$ is abelian.

If $A/\omega(A)$ is abelian, then

$$A' \subseteq \omega(A) \subseteq Z_2(A)$$

(by Theorem 2.1.16) and there is nothing more to prove. Therefore suppose that $A/\omega(A)$ is Hamiltonian. Let H be defined by $H/\omega(A) = (A/\omega(A))'$. Then

$$|H/\omega(A)| = 2,$$

and all squares in $A/\omega(A)$ lie in $H/\omega(A)$. Also $A/\omega(A)$ is generated by elements of order 4, say

$$A = \omega(A)\langle u_1, u_2, \ldots, u_r\rangle$$

where $u_i^2 \in H$ but $u_i^2 \notin \omega(A)$ for all $1 \leq i \leq r$.

Hence $H = \omega(A)\langle u_i^2\rangle$ for $1 \leq i \leq r$. So

$$[H, u_i] = [\omega(A), u_i] \subseteq Z(A)$$

for $1 \leq i \leq r$. Also $[H, \omega(A)] \subseteq Z(A)$ and so it follows that $[H, A] \subseteq Z(A)$. Therefore

$$A' \subseteq H \subseteq Z_2(A).$$

Thus A has nilpotency class at most three. $\square$

Note that groups of nilpotency class at most two are all in $\mathcal{W}_2$. The inclusion of the case $p = 2$ here is for completeness: from now on our groups will mainly involve only odd primes.

Definition 3.2.2 *For a p-group A, $e(A)$ is defined by: $p^{e(A)}$ is the exponent of $A/\gamma_3(A)$.*

The next result, a precise statement of the result of Ormerod [12] mentioned in the introduction, will be needed again in chapter six. To formulate it we need the concept of second nilpotent product of groups.

Nilpotent Products Let F be the free product of two groups A and B, let k be any natural number and let $\gamma_k(F)$ be the k-th term of the lower central series of F. The *k-th nilpotent product* of A and B is the factor group $F/(\gamma_k(F) \cap [A, B])$ and is denoted by $A *_{N_k} B$.

We also define $L_n(p^r)$ to be the free group[1] of rank n in the variety of all groups of nilpotency class at most two and exponent dividing p^r.

Definition 3.2.3 *The group*

$$G_n(p^r) = H(p, r) *_{N_2} L_n(p^r)$$

for $n \geq 1$, $r \geq 1$ is the second nilpotent product of the group $H(p, r)$ constructed in section 3.1 and $L_n(p^r)$.

Theorem 3.2.4 [Theorem A, Ormerod [12]] *Let $p > 3$ be a prime. For all $n \geq 1$, $r \geq 1$, $G_n(p^r) \in \mathcal{W}_2$.*

Conversely, if $G \in \mathcal{W}_2$ is a p-group, if $e(G) = r$, and if G can be generated by $n + 2$ elements, then G is a homomorphic image of $G_n(p^r)$.

Proof That every group $G_n(p^r)$ lies in $\mathcal{W}_2$ is proved by Ormerod: see Theorem 4.5 of [12].

Conversely suppose that $G \in \mathcal{W}_2$. By Theorem 3.2.1 G has nilpotency class at most 3. We may as well suppose that G has nilpotency class three exactly,

[1] The definition of relatively free groups and their basic properties can be found in section 2.3 of Robinson [14]

since otherwise the claim is immediate: G would have nilpotency class at most two and exponent p^r, so it would certainly be a homomorphic image of $L_n(p^r)$ and therefore of $G_n(p^r)$. Ormerod proves that G has, for some integer $n \geq 0$, generators $x_1, x_2, x_3, \ldots, x_n, x_{n+1}, x_{n+2}$ such that, for some $t \geq 1$,

$$[x_1, x_2, x_1] = x_1^{p^t}, [x_1, x_2, x_2] = x_2^{p^t}, [x_1, x_2, x_k] = x_k^{p^t} = 1, k = 3, \ldots, n+2 \quad (3.2)$$

and $[x_k, x_j] \in Z(G)$ whenever at least one of j and k belong in $\{3, \ldots, n+2\}$. It follows from these relations that, for all i, j, $[x_i, x_j]^{p^t} = 1$ so G' has exponent dividing p^t.

From the proof of Theorem 4.5 of Ormerod [12], we can choose the generators x_i in such a way that $\langle x_1 \rangle \cap \langle x_2 \rangle = 1$.

For $3 \leq j \leq n+2$, $x_i \in Z_2(G)$. Since G has nilpotency class three, at least one of x_1 and x_2, therefore, does not belong to $\omega(G)$, by Theorem 2.1.16. Since $G/\gamma_3(G)$ is a regular group, generated by the elements $x_i\gamma_3(G)$ $(1 \leq i \leq n+2)$ all of which have order dividing p^t, therefore $G/\gamma_3(G)$ has exponent dividing p^t.

Since G has nilpotency class three, at least one of $[x_1, x_2, x_1]$ and $[x_1, x_2, x_2]$ is non-trivial. Now if $[x_1, x_2, x_1] \neq 1$, we claim that $x_1\gamma_3(G)$ has order exactly p^t. Suppose, if possible, that $x_1\gamma_3(G)$ has order p^s for $s < t$. This means that $x_1^{p^s}$ belongs to $\gamma_3(G)$. Therefore there exist integers m_1 and m_2 such that

$$x_1^{p^s} = [x_1, x_2, x_1]^{m_1}[x_1, x_2, x_2]^{m_2} = x_1^{m_1 p^t} x_2^{m_2 p^t}.$$

This implies that

$$x_1^{p^s - m_1 p^t} = x_2^{m_2 p^t} \in \langle x_1 \rangle \cap \langle x_2 \rangle = 1.$$

Since $x_1^{p^t} = [x_1, x_2, x_1]$, we have the order of x_1 greater than p^t and therefore we can say that p^{t+1} divides $p^s - m_1 p^t$ which is not possible for $s < t$. Thus t is the least positive integer such that $x_1^{p^t} \in \gamma_3(G)$. Similarly if $[x_1, x_2, x_2] \neq 1$, we can show that t is the least positive integer such that $x_2^{p^t} \in \gamma_3(G)$. This means that $G/\gamma_3(G)$ has exponent p^t. This shows that $t = r = e(G)$ and hence from (3.2), we have that

$$1 = [x_1^{p^r}, x_2] = [x_1, x_2]^{p^r},$$

by Theorem 2.1.6. Therefore $x_1^{p^{2r}} = [x_1, x_2, x_1]^{p^r} = [[x_1, x_2]^{p^r}, x_1] = 1$, and similarly $x_2^{p^{2r}} = 1$.

It follows that x_1, x_2 satisfy the defining relations satisfied by the generators x, y of $H(p,r)$. Moreover $\langle x_3, \ldots, x_{n+2}\rangle$ is a group of nilpotency class at most two and exponent dividing p^r; and finally $[x_i, x_j, x_k] = 1$ whenever one of i, j, k is 1 or 2 and one is ≥ 3. Therefore there is a homomorphism from $G_n(p^r)$ onto G. □

3.3 Some properties of p-groups in $\mathcal{W}_2$

In this short section we investigate connections between two numerical invariants of p-groups in $\mathcal{W}_2$. These will be used later in chapter 6.

For convenience we write $G = G_n(p^r)$, $H = H(p,r)$ and $L = L_n(p^r)$ in what follows, p, n and r being understood, and of course $e(H) = r$.

Lemma 3.3.1 $Z_2(G) = Z_2(H)L[L,H]$.

Proof By definition of second nilpotent product, we have $G = HL[L,H]$. Also $Z_2(H)L[L,H] \subseteq Z_2(G)$; and $Z_2(G) \cap H \subseteq Z_2(H)$. Therefore

$$Z_2(H)L[L,H] \subseteq Z_2(G) \subseteq L[L,H](Z_2(G) \cap H) \subseteq L[L,H]Z_2(H)$$

which gives the result claimed. □

Lemma 3.3.2 $e(G) = e(H)$.

Proof This is because every commutator of weight three in G is a power of one of the forms $[h_1, h_2, h_3], [h_1, l_1, h_2], [h_1, l_1, l_2]$ or $[l_1, l_2, l_3]$, where $h_i \in H$, $l_i \in L (1 \leq i \leq 3)$. Here we use the Jacobi identity which holds in a metabelian group. However, all but the first of these are necessarily trivial in G. Hence $\gamma_3(G) = \gamma_3(H)$. It follows that $G/\gamma_3(G) \cong H/\gamma_3(H) *_{N_2} L$. Both factors on the right have exponent dividing $p^{e(H)}$ and $H/\gamma_3(H)$ has exactly this exponent, so, by regularity, $G/\gamma_3(G)$ has the exponent exactly $p^{e(H)}$. That is $e(G) = e(H)$, as required. □

Lemma 3.3.3 *Let A be a p-group, with $p > 3$, and Wielandt length two. Then $Z_2(A)$ has exponent dividing $p^{e(A)}$.*

Proof If A has nilpotency class at most two, there is nothing to prove. So suppose that A has nilpotency class three, and that it can be generated by $n+2$ elements. Then, for some $N \lhd G = G_n(p^{e(A)}), G/N \cong A$.

Suppose that $g \in G$ and $gN \in Z_2(G/N)$. Since $Z_2(G)N/N \subseteq Z_2(G/N)$ and since $Z_2(G)$ has exponent $p^{e(A)}$ by Theorem 3.1.2(3) and Lemma 3.3.1, we may suppose that $g \notin Z_2(G)$ and therefore that $g \in H$ but $g \notin H'$, since G is regular.

Moreover we may suppose that $g = x^m y^n$ for some integers m, n. Then for $r = e(A)$:

$$x^{mp^r} = [x, y, x]^m = [g, y, x] \in N$$

and

$$y^{np^r} = [x, y, y]^n = [y, x, y]^{-n} = [g, x, y]^{-1} \in N.$$

From this we see, using Theorem 2.1.14, that $g^{p^r} = (x^m y^n)^{p^r} = x^{mp^r} y^{np^r} \in N$.

Hence $Z_2(A)$ has exponent dividing $p^r = p^{e(A)}$. □

Lemma 3.3.4 *Let $p > 3$ be a prime, and let G_1 be a group of Wielandt length two and nilpotency class three. Also let G_2 be a p-group of nilpotency class at most two. Then $W = G_1 *_{N_2} G_2 \in \mathcal{W}_2$ if and only if the exponent of G_2 divides $p^{e(G_1)}$.*

Proof First suppose that G_2 has exponent dividing p^r where $r = e(G_1)$. Also suppose that G_2 is generated by m elements. By Theorem 3.2.4, for some positive integer n, there is an onto homomorphism $\theta : G_n(p^r) \longrightarrow G_1$. It follows that θ may be extended to an onto homomorphism $G_{m+n}(p^r) \to G_1 *_{N_2} G_2$, so $W \in \mathcal{W}_2$ by Theorem 3.2.4, as required.

Conversely suppose that $W \in \mathcal{W}_2$. Then, by Lemma 3.3.3, $Z_2(W)$ has exponent dividing $p^{e(W)}$. However, by Lemma 3.3.1, the exponent of G_2 then divides $p^{e(W)}$ which is equal to $p^{e(G_1)}$ by Lemma 3.3.2. □

Corollary 3.3.5 *With the same conditions on G_1, G_2 as in Lemma 3.3.4 let $N \subseteq [G_1, G_2]$ be a normal subgroup of W. Then $W/N \in \mathcal{W}_2$ if and only if the exponent of G_2 divides $p^{e(G_1)}$.*

Proof If the exponent of G_2 divides $p^{e(G_1)}$ then $W \in \mathcal{W}_2$ by Lemma 3.3.4 and therefore $W/N \in \mathcal{W}_2$.

Conversely suppose that $W/N \in \mathcal{W}_2$. Then

$$G_2 \cong G_2N/N \subseteq Z_2(W)N/N \subseteq Z_2(W/N)$$

and so the exponent of G_2 divides $p^{e(W/N)}$ by Lemma 3.3.3. Therefore

$$(W/N)/\gamma_3(W/N) \cong W/\gamma_3(W)N \cong (W/\gamma_3(W))/(\gamma_3(W)N/\gamma_3(W))$$

so $e(W/N) \leq e(W) = e(G_1)$. Therefore the exponent of G_2 divides $p^{e(G_1)}$, as required. □

Chapter 4

Wielandt length of a supersoluble group

In this chapter we develop a technique for calculating the Wielandt length of a semidirect of two groups of coprime order. Every supersoluble group has this structure: if G is a supersoluble group and p is the largest prime dividing $|G|$, then a Sylow p-subgroup is normal. Therefore if π is a set of primes for which the Sylow p-subgroups of a supersoluble group G are normal, and A is the Hall π-subgroup of G and B is a Hall π'-subgroup of G, then $G = BA$ and we are able to calculate the Wielandt subgroup and the Wielandt length of G.

We use this technique in section 1 to prove, among other things, that if a Sylow p-subgroup of a supersoluble group G has Wielandt length $n > 1$, then a Sylow p-subgroup of $G/\omega(G)$ has Wielandt length at most $n - 1$.

In section 2 we establish a relation between the Wielandt length of a supersoluble group and various invariants of its Sylow subgroups. We prove that if all Sylow subgroups of G have Wielandt length at most n, then G has Wielandt length at most $n + 1$ where $n = 1, 2$. A general result seems to require more information on the Wielandt structure of nilpotent groups than is presently available. However we manage to prove that if G is a supersoluble group and n is the maximum of the nilpotency classes of the Sylow subgroups of G, then G has Wielandt length at most $n + 1$, for all n.

An example is given in section 3 which shows that the bound given in section 2 is best possible. In section 2 we give an example to prove that this bound on the Wielandt length of supersoluble groups is not valid for soluble groups in general.

4.1 The Wielandt subgroup

We begin with a theorem which proves very useful in calculating the Wielandt subgroup of the semidirect product of two groups of coprime order. To prove this theorem we need the following three lemmas.

Lemma 4.1.1 *Let $G = BA$ be a semidirect product of subgroups A, B of coprime order, with A normal in G. Then for each subgroup H of G, there exists $g \in G$ such that*

$$H^g = (H^g \cap B)(H^g \cap A).$$

Proof We have

$$HA/A \cong H/H \cap A.$$

Since $(|H \cap A|, |H/H \cap A|) = 1$, H has a subgroup C isomorphic to $H/H \cap A$ so that $H = C(H \cap A)$, using the Schur-Zassenhaus theorem. Also, by Theorem 2.1.7,

$$HA = (HA \cap B)A,$$

and

$$HA \cap B \cong HA/A \cong H/(H \cap A) \cong C.$$

It follows that $HA \cap B$ and C are both Hall subgroups of HA for the same set of primes, and hence are conjugate in HA. Therefore, for some $g \in HA$, $C^g \subseteq B$. Also

$$H^g = C^g(H^g \cap A).$$

Now again, by Theorem 2.1.7, we have

$$H^g \cap B = C^g(H^g \cap A) \cap B = C^g$$

and thus

$$H^g = (H^g \cap B)(H^g \cap A).$$

□

Corollary 4.1.2 *Let $G = BA$ be a semidirect product of subgroups A, B of coprime order, with A normal in G. Then, for each subnormal subgroup S of G,*

$$S = (S \cap B)(S \cap A).$$

Proof We use induction on the defect s of S in G.

First suppose that $s = 1$. This means S is normal in G and therefore by using Lemma 4.1.1, for some $g \in G$,

$$S = S^g = (S^g \cap B)(S^g \cap A) = (S \cap B)(S \cap A),$$

as required.

If the defect of S in G is $s > 1$, suppose the result is already proved for subnormal subgroups of defect less than s. Since S is subnormal, it is contained in a normal subgroup S_0 of G and has defect $s - 1$ in S_0. By the case we have already done,

$$S_0 = (S_0 \cap B)(S_0 \cap A).$$

Hence S_0 satisfies the hypothesis of the theorem. Therefore, by induction,

$$S = (S \cap S_0 \cap B)(S \cap S_0 \cap A) = (S \cap A)(S \cap B),$$

as required. This completes the induction. □

Lemma 4.1.3 *Let $G = BA$ be a semidirect product of subgroups A and B with A normal in G. Then*

$$\omega(G) \cap B \subseteq \omega(B).$$

Proof If S is any subnormal subgroup of B, then SA is subnormal in G. Therefore for each $x \in \omega(G) \cap B$, $(SA)^x = SA$ which implies that $S^x A = SA$. Now for all $s \in S$ there is some $s_1 \in S$ so that $s^x \in s_1 A$ and so $s_1^{-1} s^x \in A \cap B = 1$ so that $s^x = s_1 \in S$, and this for all $x \in \omega(G) \cap B$. Hence we get that

$$\omega(G) \cap B \subseteq \omega(B).$$

□

Lemma 4.1.4 *Let $G = BA$ be a semidirect product of subgroups A, B of coprime order, with A normal in G. Then*

$$\omega(G) \cap A = \omega(A).$$

Proof First of all we claim that $\omega(A) \subseteq \omega(G)$. To prove this, let Ω denote the class of groups G where $G = BA$ with A, B subgroups of coprime order and A normal. Now suppose that for some $G \in \Omega$, if possible, S is a subnormal subgroup of G so that $\omega(A) \not\subseteq N_G(S)$. Let Γ be the subclass of Ω of those G which contain such an S with $|S|$ as small as possible. Now assume that G is a minimal member of Γ.

We claim that $G = S\omega(A)$. To prove this, suppose that, if possible, $S\omega(A) \neq G$ and put $G^* = S\omega(A)$ a subnormal subgroup of G. By Corollary 4.1.2 we can write

$$G^* = (S \cap B)(S \cap A)\omega(A) = B^* A^*$$

where $B^* = S \cap B$ and $A^* = (S \cap A)\omega(A)$. It is easy to see that $A^* = G^* \cap A$ is normal in G^* with A^* and B^* coprime in order, so that G^* belongs to Ω. Every subnormal subgroup of A^* is subnormal in A. Therefore

$$\omega(A) \subseteq \omega(A^*);$$

and

$$\omega(A^*) \subseteq \omega(G^*),$$

by the minimality of G. Therefore S is not normalised by $\omega(A^*)$. But we know that $|S|$ is as small as possible and thus G^* belongs to Γ. But then $G^* \neq G$ contradicts the minimality of G, so $\omega(A^*)$, and hence $\omega(A)$, is contained in $N_G(S)$, a contradiction. This proves our claim that $G = G^* = S\omega(A)$.

Let N be a non-trivial normal subgroup of S. Of course N is subnormal in G. If $N \neq S$, then N is normalised by $\omega(A)$ by minimality of S, so N is normal in G and hence S/N is subnormal in G/N, a group in Ω. Again, by minimality of S,

$$\omega(A)N/N \subseteq \omega(AN/N) \subseteq N_{G/N}(S/N)$$

which implies that $\omega(A) \subseteq N_G(S)$, a contradiction to our supposition that $\omega(A)$ does not normalise S. Hence $N = S$. This implies that S is simple. Hence

$$\omega(A) \cap S = 1.$$

Now since S is subnormal in G, there exists some positive integer n such that

$$[\omega(A), nS] = S \cap \omega(A) = 1.$$

However $S \not\subseteq A$ or else S sn A and so $\omega(A) \subseteq N_G(S)$, a contradiction. Therefore there is a prime q dividing $|S|$ which does not divide $|A|$. Let Q be a Sylow q-subgroup of S. Then

$$[\omega(A), nQ] = 1.$$

Since $(|Q|, |\omega(A)|) = 1$, it follows from Lemma 2.1.10 (2), that $[\omega(A), Q] = 1$. This means $1 \neq C_S(\omega(A)) \lhd S$. But S is simple, so $C_S(\omega(A)) = S$, contradicting the fact that $\omega(A)$ does not normalise S. Hence the class Γ is empty and we have proved that, for all groups G satisfying the hypothesis of the lemma, $\omega(A) \subseteq \omega(G)$.

Thus $\omega(A) \subseteq \omega(G) \cap A$. But since A is normal in G, every subnormal subgroup of A is subnormal in G, so $\omega(G) \cap A \subseteq \omega(A)$. Thus $\omega(G) \cap A = \omega(A)$, completing the proof of the lemma. □

Now we use all these results to prove the following theorem.

Theorem 4.1.5 *Let $G = BA$ be a semidirect product of subgroups A, B of coprime order with A nilpotent and normal. Also suppose that G is soluble. If P is the set of those elements of $\omega(B)$ which act by conjugation as power automorphisms on A, then*

$$\omega(G) = P\omega(A).$$

Proof Suppose that G satisfies the hypothesis of the theorem so that $G = BA$, $(|B|, |A|) = 1$, and A is normal and nilpotent in G. Using Corollary 4.1.2 and Lemmas 4.1.3 and 4.1.4, we get

$$\omega(G) = (\omega(G) \cap B)\omega(A) \subseteq \omega(B)\omega(A).$$

Since A is normal and nilpotent, its subgroups are subnormal in A and hence subnormal in G. Therefore $\omega(G) \cap B$ normalises all subgroups of A and so we have $\omega(G) \cap B \subseteq P$.

We claim that

$$P \subseteq \omega(G) \cap B.$$

By definition $P \subseteq B$ and we only need to show that $P \subseteq \omega(G)$.

To prove this, let S be an arbitrary subnormal subgroup of G. By Corollary 4.1.2, we have

$$S = (S \cap B)(S \cap A).$$

Since $S \cap B$ is subnormal in B, it is normalised by P. It follows from the definition of P that P normalises $S \cap A$ and hence S. This proves our claim that

$$P \subseteq \omega(G) \cap B.$$

Thus

$$\omega(G) = P\omega(A),$$

as required. □

By Corollary 2.3.3 we see that if A is the subgroup generated by all normal Sylow subgroups of a supersoluble group G, then A is non-trivial and complemented.

The following is an immediate corollary of Theorem 4.1.5 and gives the information we will need about the Wielandt subgroup of a supersoluble group.

Corollary 4.1.6 *Let π be the set of all primes for which the Sylow p-subgroups of a supersoluble group G are normal. Let A be the Hall π-subgroup and B a Hall π'-subgroup of G, so that A is normal in G and $G = BA$.*

If P is the set of those elements of $\omega(B)$ which act by conjugation as power automorphisms on A, then

$$\omega(G) = P\omega(A).$$

□

Now we prove a result which gives information about the Sylow p-subgroups of the Wielandt subgroup of a supersoluble group.

Lemma 4.1.7 *If A is a Sylow p-subgroup of a supersoluble group G, then*

$$\omega(A) \cap F(G) = O_p(\omega(G)).$$

Proof Let $Q = O_{p'}(G)$. Then,

$$(\omega(A) \cap F(G))Q/Q \subseteq \omega(A)Q/Q.$$

Now since G/Q is supersoluble, it has a normal Sylow subgroup which is a p-group, because $F(G/Q)$ is a p-group by Theorem 2.3.2. Thus AQ/Q, being a Sylow p-subgroup of G/Q, is normal. Also $\omega(A)Q/Q \subseteq \omega(AQ/Q)$. Again since AQ/Q is a normal Hall subgroup of G/Q, by Lemma 4.1.4 we have $\omega(AQ/Q) \subseteq \omega(G/Q)$ and so

$$(\omega(A) \cap F(G))Q/Q \subseteq \omega(G/Q).$$

Now by Theorem 2.2.9, the local Wielandt subgroup of G/Q is given by $\omega(G/Q) = \omega^p(G)/Q$ and therefore we get

$$(\omega(A) \cap F(G))Q/Q \subseteq \omega^p(G)/Q$$

and so

$$\omega(A) \cap F(G) \subseteq \omega^p(G).$$

Since $\omega(A) \cap F(G)$ is a subnormal p-subgroup of G,

$$\omega(A) \cap F(G) \subseteq O_{q'}(G)$$

for all primes $q \neq p$. But by Theorem 2.2.7, we have $O_{q'}(G) \subseteq \omega^q(G)$. Therefore, using Theorem 2.2.8, we get

$$\omega(A) \cap F(G) \subseteq \cap_r \omega^r(G) = \omega(G)$$

where the intersection is over all primes r. It follows that, since $\omega(A) \cap F(G)$ is a p-group and subnormal,

$$\omega(A) \cap F(G) \subseteq O_p(\omega(G)).$$

In the opposite direction we claim that $O_p(\omega(G)) \subseteq \omega(A)$. To prove this claim suppose that S is a subnormal subgroup of A. Then SQ is subnormal in G.

Therefore for $x \in O_p(\omega(G)) \subseteq A$, we have $(SQ)^x = SQ$. This implies $S^xQ = SQ$. Arguing as in Lemma 4.1.3, we get $S^x = S$ and therefore $O_p(\omega(G)) \subseteq \omega(A)$. Now $O_p(\omega(G))$ is a nilpotent characteristic subgroup of $\omega(G)$ therefore normal in G and so it is contained in $F(G)$. Thus we have $O_p(\omega(G)) \subseteq \omega(A) \cap F(G)$. This means, therefore, that $\omega(A) \cap F(G) = O_p(\omega(G))$ as required. □

The following is an easy corollary of the above lemma.

Corollary 4.1.8 *Let G be a supersoluble group and suppose a Sylow p-subgroup of G has nilpotency class $n > 1$. Then a Sylow p-subgroup of $G/\omega(G)$ has nilpotency class at most $n - 1$.*

Proof Let A be a Sylow p-subgroup of G. Since A has nilpotency class n, $\gamma_n(A) \subseteq Z(A) \subseteq \omega(A)$. Also $\gamma_n(A) \subseteq A' \subseteq G' \subseteq F(G)$ by Theorem 2.3.1, and since $n > 1$. This means $\gamma_n(A) \subseteq F(G) \cap \omega(A)$. But then, by Lemma 4.1.7, $\gamma_n(A) \subseteq \omega(G)$.

Since $A\omega(G)/\omega(G)$ is a Sylow p-subgroup of $G/\omega(G)$ and

$$\gamma_n(A\omega(G)/\omega(G)) = \gamma_n(A)\omega(G)/\omega(G) = 1,$$

$A\omega(G)/\omega(G)$ has nilpotency class at most $n - 1$. □

4.2 More on supersoluble groups

In this section our aim is to relate the Wielandt length of a supersoluble group with other invariants and particularly with invariants of its Sylow subgroups. It would be very nice if we could find a relation between the Wielandt length of a supersoluble group and the Wielandt length of its Sylow subgroups. We have been able to do so when the Wielandt length of Sylow subgroups is at most two, but a general result seems difficult. We can, however, bound the Wielandt length in terms of the nilpotency classes of the Sylow subgroups. We begin with the following lemma.

Lemma 4.2.1 *Let G be a supersoluble group. If all Sylow p-subgroups of G are abelian except for $p = 2$, and if the Sylow 2-subgroups have class at most two, then G has Wielandt length at most two.*

Proof We begin by supposing that, for some prime p dividing $|G|$, $O_{p'}(G) = 1$. In this case $O_p(G)$ is a normal Sylow p-subgroup of G, by Theorem 2.3.2.

Consider the case $p \neq 2$. Then, by hypothesis, $O_p(G)$ is abelian and so, by Lemma 4.1.4, we have $O_p(G) \subseteq \omega(G)$. Moreover $O_p(G) = F(G)$ and using Theorem 2.3.1, $G/O_p(G)$ is abelian. It follows that $G/\omega(G)$ is abelian and $\omega_2(G) = G$. That is G has Wielandt length at most two.

Now consider the case when $p = 2$. In this case $O_p(G) = G$ using Theorem 2.3.2 and so G has nilpotency class at most two. But $Z(G) \subseteq \omega(G)$ and hence $G/\omega(G)$ is again abelian. Thus, under the assumption $O_{p'}(G) = 1$, G has Wielandt length at most two.

Now consider the general case and, for some prime p dividing $|G|$, put $H = G/O_{p'}(G)$. Note that $O_{p'}(H) = 1$. Using Theorems 2.2.7 and 2.2.9, the local Wielandt subgroup of H is given as follows:

$$\omega^p(H) = \omega^p(G/O_{p'}(G)) = \omega^p(G)/O_{p'}(G) = \omega(G/O_{p'}(G)) = \omega(H).$$

From the above case $H/\omega(H)$ is abelian and so is $H/\omega^p(H)$. Now

$$H/\omega^p(H) = (G/O_{p'}(G))/(\omega^p(G/O_{p'}(G)) = (G/O_{p'}(G))/(\omega^p(G)/O_{p'}(G))$$

which is isomorphic to $G/\omega^p(G)$. Thus $G/\omega^p(G)$ is abelian. This implies that $G' \subseteq \omega^p(G)$ for all p dividing $|G|$. Hence $G' \subseteq \cap_p \omega^p(G)$ and so, by Theorem 2.2.8, $G' \subseteq \omega(G)$ and thus $G/\omega(G)$ is abelian. This shows G has Wielandt length at most two. □

Using this lemma we prove the following theorem.

Theorem 4.2.2 *Let G be a supersoluble group. If all Sylow subgroups of G have Wielandt length at most n, then G has Wielandt length at most $n + 1$, where $n = 1, 2$.*

Proof First suppose that all Sylow p-subgroups of G have Wielandt length one. This means that each of them is either abelian or Hamiltonian. More precisely all Sylow p-subgroups of G are abelian except the Sylow 2-subgroups which could be Hamiltonian and hence have class at most two. This means, by Lemma 4.2.1, that G has Wielandt length at most two.

Now consider the case when all Sylow p-subgroups of G have Wielandt length at most two. Let A be a Sylow p-subgroup of G. By Lemma 3.2.1, A has nilpotency class at most three. Thus all Sylow p-subgroups of G have nilpotency class at most three. This implies, from Corollary 4.1.8, that the nilpotency class of each of the Sylow p-subgroups of $G/\omega(G)$ is at most two.

Let A be any non-abelian Sylow p-subgroup of G. Then A has Wielandt length at most two. If $p \neq 2$, then $A/\omega(A)$ is abelian. Also, since G is supersoluble, $G' \subseteq F(G)$ by Theorem 2.3.1, and so

$$A' \subseteq \omega(A) \cap G' \subseteq \omega(A) \cap F(G).$$

But, by Lemma 4.1.7, $\omega(A) \cap F(G) = O_p(\omega(G))$, and therefore

$$A' \subseteq O_p(\omega(G)) \subseteq \omega(G).$$

This means that $A\omega(G)/\omega(G) \cong A/(A \cap \omega(G))$ is abelian and thus we can say that all Sylow p-subgroups of $G/\omega(G)$ are abelian except possibly for the Sylow 2-subgroups, and we proved above that their nilpotency class is at most two. Thus using Lemma 4.2.1 again, we conclude that $G/\omega(G)$ has Wielandt length two, and hence G has Wielandt length at most three. □

To find similar result for groups whose Sylow subgroups have greater Wielandt length seems difficult because more information is required about the Wielandt structure of nilpotent groups.

We show next that the Wielandt length of a supersoluble group is bounded by the nilpotency class of its Sylow subgroups. In the next section we give an example to show this bound is best possible.

Theorem 4.2.3 *If G is a supersoluble group and n is the maximum of the nilpotency classes of the Sylow subgroups of G, then $G \in \mathcal{W}_{n+1}$.*

Proof We prove this by induction on n. The case $n = 1$ follows immediately from Lemma 4.2.1.

Suppose the theorem holds for $n = k \geq 1$ so that if all Sylow subgroups of G have nilpotency class at most k, then $G \in \mathcal{W}_{k+1}$.

Consider the case when $n = k + 1$, that is, when all Sylow subgroups of G have nilpotency class at most $k + 1$. By Corollary 4.1.8, this implies that all Sylow subgroups of $G/\omega(G)$ have nilpotency class at most k. By our supposition $G/\omega(G) \in \mathcal{W}_{k+1}$ or, in other words, $\omega_{k+1}(G/\omega(G)) = G/\omega(G)$. But by definition $\omega_{k+1}(G/\omega(G)) = \omega_{k+2}(G)/\omega(G)$. Therefore $\omega_{k+2}(G) = G$. Thus we conclude that $G \in \mathcal{W}_{k+2}$. This means that the theorem holds for $n = k + 1$ and therefore, by induction, for all n. □

Note that supersolubility of G is necessary for this result. The following example shows that in general no such bound is possible. Let $p_1, p_2, p_3, \ldots, p_i, \ldots$ be infinitely many distinct primes and C_i be a cyclic group of order p_i for $i \geq 1$. Define $G_1 = C_1$ and, once G_{n-1} is defined, put

$$G_n = C_n wr G_{n-1}.$$

It is obvious that G_n has all its Sylow subgroups abelian and that G_1 has Wielandt length one. We prove by induction on n that G_n has Wielandt length n. Suppose that $n \geq 2$ and that G_{n-1} has Wielandt length $n - 1$. Now consider $G_n = C_n wr G_{n-1} = G_{n-1}B$, where B is the base group of G_n. By Theorem 4.1.5, $\omega(G_n) = P\omega(B)$ where P is the set of those elements of $\omega(G_{n-1})$ which act as power automorphisms on B. But no non-trivial element of G_{n-1} acts as a power automorphism on B. So $P = 1$ and $\omega(G_n) = \omega(B) = B$, as B is abelian. This implies $G_n/\omega(G_n) \cong G_{n-1}$ which, by our supposition, has Wielandt length $n - 1$. Therefore, by induction, G_n has Wielandt length n.

4.3 An example

In this section we give an example to show that the bound in Theorem 4.2.3 is best possible. But first we need the following important Lemma.

Lemma 4.3.1 *a) For each positive integer n and each prime $p \geq 5$, there is a relatively free group A of exponent p and nilpotency class exactly n.*

b) For the group A in (a), we have

$$A' = Z_{n-1}(A).$$

Proof a) By Theorem 2.1.8, there exists an insoluble, locally finite group G of exponent p. Let F be the free group of countable rank in the variety generated by G. Then F is locally finite, insoluble and therefore non-nilpotent. Let X be a free generating set of F. It follows that there is a subset $\{x_1, x_2, \ldots, x_m\}$ of X of cardinality $m \geq n$, such that

$$F_0 = \langle x_1, x_2, \ldots, x_m \rangle$$

has nilpotency class at least n. Were this not so, F would be nilpotent. Of course F_0 is finite, and relatively free on $\{x_1, x_2, \ldots, x_m\}$. Write

$$A = F_0/\gamma_{n+1}(F_0)$$

and

$$y_i = x_i\gamma_{n+1}(F_0)$$

for $1 \leq i \leq m$, so that A is relatively free on $\{y_1, y_2, \ldots, y_m\}$, of exponent p, and of nilpotency class n exactly.

b) Of course $A' \subseteq Z_{n-1}(A)$. If $A' \neq Z_{n-1}(A)$ choose $z_1 \in Z_{n-1}(A)$ such that $z_1 \notin A'$. Then, by Burnside's Basis Theorem, there is a generating set $\{z_1, z_2, \ldots, z_m\}$ of A. Moreover this is a free generating set, because there is an endomorphism ϕ of A determined by

$$\phi : y_i \mapsto z_i$$

for $1 \leq i \leq m$. This is onto and therefore one-to-one as A is finite.

Now let $a_1, a_2, \ldots, a_n$ be arbitrary elements of A. Then there is an endomorphism ψ of A such that

$$\psi(z_i) = a_i, \psi(z_j) = 1$$

for $1 \leq i \leq n$ and $n+1 \leq j \leq m$. Hence, since $z_1 \in Z_{n-1}(A)$,

$$1 = \psi(1) = \psi([z_1, z_2, \ldots, z_n]) = [a_1, a_2, \ldots, a_n].$$

Therefore, since every simple commutator in weight n in A is trivial, A has nilpotency class at most $n-1$, a contradiction. Hence $A' = Z_{n-1}(A)$, as required.

□

Example 4.3.2 Let χ be the mapping defined on A of Lemma 4.3.1 by $\chi(y_1) = (y_1)^{-1}$ and $\chi(y_i) = y_i$ for $i = 2, 3, \ldots, m$. Since $y_1, y_2, \ldots, y_m$ are free generators of A, χ can be extended to an automorphism of A. Clearly χ does not act as a power automorphism on A/A'. Put $B = \langle \chi \rangle$ and $G = BA$ for the semidirect product. Obviously G is supersoluble and it follows immediately from the following lemma that G has Wielandt length $n+1$.

Lemma 4.3.3 *Let A be a p-group of exponent p and nilpotency class n with p an odd prime and b be an automorphism of A with $|b| = 2$ such that b does not act as a power automorphism on A/A'. Also suppose that $A' = Z_{n-1}(A)$.*

Put $B = \langle b \rangle$ and $G = BA$, the semidirect product. Then G has Wielandt length $n+1$.

Proof Since G is supersoluble we have by Corollary 4.1.6, that $\omega(G) = P\omega(A)$ where P is the set of those elements in B which act as a power automorphism on A. But by the definition of b, we have $P = 1$. Also since A is a p-group of exponent p, every element of $\omega(A)$ centralises every element of A. Hence $\omega(A) = Z(A)$ and thus $\omega(G) = Z(A)$.

We prove by induction on n that G has Wielandt length $n+1$. For $n = 1$, we have $Z(A) = A$ and so $\omega(G) = A$. So $G/\omega(G) \cong B$ and thus G has Wielandt length two.

Suppose that A has nilpotency class $n > 1$ and, if A has nilpotency class $n-1$, that G has Wielandt length n. Then $G/\omega(G) \cong BA/Z(A)$. Here we know that $A/Z(A)$ has nilpotency class $n-1$ and exponent p. Also b does not act as a power automorphism on $(A/Z(A))/(A/Z(A))' \cong A/A'$ since $Z(A) \subseteq A'$ by hypothesis. Therefore $G/\omega(G)$ has Wielandt length n and thus G has Wielandt length $n+1$.

□

Chapter 5

On the Wielandt length of subgroups

5.1 Introduction

We know that all homomorphic images and subnormal subgroups of a finite group G have Wielandt length at most the Wielandt length of G. However, in general, the Wielandt length of a subgroup of G is not bounded by the Wielandt length of G. This is illustrated by the following example given in Section 6 of Cossey [7]. Let G be the semidirect product of

$$N = \langle a, b | a^{2^n} = b^{2^n} = 1, [a, b] = 1 \rangle,$$

the direct product of two cyclic groups of order 2^n, by

$$H = \langle x, y | x^2 = y^3 = 1, y^x = y^{-1} \rangle,$$

the symmetric group on three elements, in such a way that

$$a^x = b, b^x = a, a^y = b, b^y = a^{-1}b^{-1}.$$

Observe that $u = ab^{-1}$ is inverted by x and so the group $\langle u, x \rangle$ is dihedral of order 2^{n+1}. The dihedral group of order 2^{n+1} has Wielandt length n because the Wielandt series coincides with the upper central series. Hence a Sylow 2-subgroup of G has Wielandt length at least n. On the other hand G itself has Wielandt length two: G has a unique chief series

$$1 \lhd N^{2^{n-1}} \lhd N^{2^{n-2}} \lhd \ldots \lhd N \lhd N\langle y \rangle \lhd G.$$

Hence every proper subnormal subgroup S of G is in $N\langle y\rangle$. By Theorem 4.1.5 $N \subseteq \omega(N\langle y\rangle)$ and hence $N \subseteq N_G(S)$. Therefore $N \subseteq \omega(G)$. Since $G/N \cong H$ is a T-group, $G \in \mathcal{W}_2$.

In this chapter we investigate the class $\mathcal{K}$ of finite soluble groups G such that each subgroup H of G has Wielandt length at most the Wielandt length of G. We conjecture that a group G belongs to $\mathcal{K}$ if and only if all Sylow subgroups of G have Wielandt length at most that of G. We denote by $\mathcal{V}$ the class of groups G whose Sylow subgroups have Wielandt length at most that of G. In other words we conjecture that $\mathcal{V} = \mathcal{K}$. It is easy to see that $\mathcal{K} \subseteq \mathcal{V}$ but the proof of the converse seems difficult.

Let $\mathcal{L}$ be the class of soluble groups G which satisfy the condition that, for each subgroup H and each normal subgroup N of G,

$$(HN/N) \cap \omega(G/N) \subseteq \omega(HN/N). \tag{5.1}$$

This inclusion is, of course, always true whenever H is subnormal in G. The main result of this chapter is that $\mathcal{L}$ is precisely the class of groups whose p-length is one for all primes p: see Theorem 5.2.5 below.

5.2 Characterising $\mathcal{L}$-groups

We begin with the following result.

Theorem 5.2.1 $\mathcal{L} \subseteq \mathcal{K}$. □

The proof is an immediate consequence of the following two lemmas.

Lemma 5.2.2 *The class $\mathcal{L}$ is closed under taking homomorphic images.*

Proof Let G be a group such that $G \in \mathcal{L}$ and M be a normal subgroup of G. We show that $G/M \in \mathcal{L}$.

Let K/M be a subgroup of G/M and N/M be a normal subgroup of G/M. We must show that

$$(K/M)(N/M)/(N/M) \cap \omega((G/M)/(N/M)) \subseteq \omega((K/M)(N/M)/(N/M)) \tag{5.2}$$

which is to say

$$(KN/M)/(N/M) \cap \omega((G/M)/(N/M)) \subseteq \omega((KN/M)/(N/M)).$$

Under the natural isomorphism $(G/M)/(N/M) \cong G/N$, the last inclusion is equivalent to

$$KN/N \cap \omega(G/N) \subseteq \omega(KN/N).$$

Since this is true because $G \in \mathcal{L}$, 5.2 is proved. Hence $\mathcal{L}$ is quotient closed. □

The next lemma gives a sufficient condition for a subgroup H of a soluble group G to have Wielandt length bounded by the Wielandt length of G.

Lemma 5.2.3 *If $G \in \mathcal{L}$ has Wielandt length n and if H is a subgroup of G, then H has Wielandt length at most n.*

Proof We prove this by induction on n. When $n = 1$, G is a T-group. Then, by Theorem 2.2.5, every subgroup of G is also a T-group, so of Wielandt length one.

Suppose inductively that if $G \in \mathcal{L}$ has Wielandt length $k \geq 1$ and H is a subgroup of G then H has Wielandt length at most k. Now suppose that $G \in \mathcal{L}$ is a group with Wielandt length $k+1$ and that H is a subgroup of G. $G/\omega(G)$ has Wielandt length k and, by Lemma 5.2.2, is in $\mathcal{L}$. It follows, by induction, that

$$H\omega(G)/\omega(G) \cong H/H \cap \omega(G)$$

has Wielandt length at most k. Now

$$H \cap \omega(G) \subseteq \omega(H),$$

which can be seen by putting $N = 1$ in (5.1). Hence $H/\omega(H)$ is a quotient of $H/H \cap \omega(G)$ and so $H/\omega(H)$ has Wielandt length at most k. Thus H has Wielandt length at most $k+1$. This completes the induction, and the proof of the lemma. □

The following example shows that not all groups belong to $\mathcal{L}$

Example 5.2.4 S_4, the symmetric group on four letters, is not in $\mathcal{L}$. Let H be a Sylow 2-subgroup of S_4. Then H is dihedral of order 8, so $|\omega(H)| = 2$. However $\omega(S_4) \cong C_2 \times C_2$, so $|\omega(S_4) \cap H| = 4$, contradicting (5.1) even with $N = 1$.

However we are able to prove that if $\mathcal{R}$ denotes the class of groups having p-length one for all primes p, then $\mathcal{R} = \mathcal{L}$. In other words we give a characterisation of the class of groups G satisfying (5.1) for all subgroups H: it is just the class of groups of p-length one for all primes p.

Theorem 5.2.5 $\mathcal{L} = \mathcal{R}$.

Proof First we prove $\mathcal{R} \subseteq \mathcal{L}$. Suppose, to the contrary, that $\mathcal{R} \nsubseteq \mathcal{L}$ and let $G \in \mathcal{R} \backslash \mathcal{L}$ be of least possible order. Then, for some subgroup H of G and some normal subgroup N of G,

$$(HN/N) \cap \omega(G/N) \nsubseteq \omega(HN/N).$$

Now $G/N \in \mathcal{R}$ and this non-inclusion shows that $G/N \notin \mathcal{L}$. Hence, by the minimality of G, $N = 1$.

Let us suppose that, for some prime p dividing $|G|$, $O_{p'}(G) = 1$. Then, since $G \in \mathcal{R}$, $G = BA$, where $A = O_p(G)$ is a Sylow p-subgroup of G, and B is a complement for A in G. By Lemma 4.1.1, for a suitable choice of B,

$$H = (H \cap B)(H \cap A).$$

Note that $H \cap A = O_p(H)$ is a Sylow p-subgroup of H. Now by Theorem 4.1.5,

$$\omega(G) = P\omega(A)$$

where $P = \omega(G) \cap B$ is the set of those elements of $\omega(B)$ which act by conjugation as power automorphisms on A. Also since $H \cap \omega(G) \lhd H$, we have from Corollary 4.1.2 that

$$H \cap \omega(G) = [(H \cap \omega(G)) \cap (H \cap B)][(H \cap \omega(G)) \cap (H \cap A)]$$

and so

$$H \cap \omega(G) = (H \cap P)(H \cap \omega(A)), \tag{5.3}$$

also using Lemma 4.1.4. Let S be an arbitrary subnormal subgroup of H. Then, by Corollary 4.1.2, we have

$$S = (S \cap (H \cap B))(S \cap (H \cap A) = (S \cap B)(S \cap A). \tag{5.4}$$

We show that $H \cap P$ normalises S. Since $B \in \mathcal{R}$ and $|B| < |G|$, $B \in \mathcal{L}$. Therefore

$$H \cap P \subseteq (H \cap B) \cap \omega(B) \subseteq \omega(H \cap B)$$

and so $H \cap P$ normalises $S \cap B$. Since $H \cap P \subseteq P$, $H \cap P$ normalises $S \cap A$ and therefore S.

Now we show that $H \cap \omega(A)$ normalises S. Note that

$$H \cap \omega(A) = (H \cap A) \cap \omega(A) \subseteq \omega(H \cap A),$$

since A is nilpotent, and therefore

$$H \cap \omega(A) \subseteq \omega(H)$$

by Lemma 4.1.4. Hence, using (5.3), $H \cap \omega(A)$ normalises S. We have shown, therefore, that $H \cap \omega(G)$ normalises every subnormal subgroup of H, so

$$H \cap \omega(G) \subseteq \omega(H).$$

This is a contradiction to our choice of G and H. Hence $O_{p'}(G) \neq 1$ for all primes p. For some prime p dividing $|G|$, write $N = O_{p'}(G)$. Using Theorems 2.2.7 and 2.2.9, we see that

$$\omega^p(G/N) = \omega^p(G)/N = \omega(G/N).$$

Hence since $|G/N| < |G|$,

$$HN/N \cap \omega^p(G/N) \subseteq \omega(HN/N)$$

and therefore, from Theorems 2.2.7 and 2.2.8,

$$HN/N \cap \omega^p(G)/N \subseteq \omega^p(HN/N).$$

From this we conclude, from Theorem 2.1.7, that

$$[H \cap \omega^p(G)]N/N \subseteq \omega^p(HN/N).$$

Via the natural homomorphism $HN/N \to H/H \cap N$ we deduce that

$$(H \cap \omega^p(G))/H \cap N \subseteq \omega^p(H/H \cap N) = \omega^p(H)/H \cap N,$$

from Theorem 2.2.7. It follows that

$$H \cap \omega^p(G) \subseteq \omega^p(H).$$

Since this is true for all primes p dividing $|G|$ we may use Theorem 2.2.8 to show that

$$H \cap \omega(G) \subseteq \omega(H).$$

This contradiction finally shows that our initial assumption was wrong. Hence $\mathcal{R} \subseteq \mathcal{L}$ as claimed.

Conversely, we prove indirectly that $\mathcal{L} \subseteq \mathcal{R}$, supposing that $\mathcal{L} \nsubseteq \mathcal{R}$ and deriving a contradiction. So suppose that $G \in \mathcal{L}$ and that $|G|$ is smallest for those G not in $\mathcal{R}$. By Lemma 5.2.2 $G/O_{p'}(G) \in \mathcal{L}$ for all primes p dividing the order of G. Now, by the minimality of G, if for some prime p dividing $|G|$, $O_{p'}(G) \neq 1$, then $G/O_{p'}(G)$ is contained in $\mathcal{R}$. This means $G/O_{p'}(G)$ has a normal Sylow p-subgroup. Thus G has p-length one. Therefore $O_{p'}(G) = 1$ for some prime p and, since $O_p(G) \neq 1$ and $G/O_p(G) \in \mathcal{L}$ is of order less than $|G|$, $G/O_p(G)$, and therefore G, has q-length one for all primes $q \neq p$. Note that $F(G) = O_p(G)$ and that G has p-length ≤ 2.

By the minimality of G, if $\Phi(G) \neq 1$ we have $G/\Phi(G) \in \mathcal{R}$ and since $\mathcal{R}$ is a saturated formation, we have $G \in \mathcal{R}$. This contradicts our supposition and so $\Phi(G) = 1$.

Now by Theorem 2.1.13, we see that $F(G)$ is the product of all minimal normal subgroups of G. If M and N are two different minimal normal subgroups of G, then by the minimality of G, we have G/M and G/N belong to $\mathcal{R}$. But since $\mathcal{R}$ is a formation, $G \in \mathcal{R}$, a contradiction. Hence G has only one minimal normal subgroup M and therefore

$$M = F(G) = O_p(G).$$

By Theorem 2.1.12, M is complemented by a maximal subgroup S of G so that $G = SM$.

Let K be a Sylow p-subgroup of S. Since $G \in \mathcal{L}$, we have

$$KM \cap \omega(G) \subseteq \omega(KM).$$

But

$$M \subseteq KM \cap \omega(G),$$

and therefore $M \subseteq \omega(KM)$. Since KM is nilpotent, K is subnormal in KM. Hence

$$[M, K] \subseteq M \cap K = 1$$

and therefore $K \subseteq C_S(M)$. If $K \neq 1$, then $C_S(M)$ is a non-trivial normal subgroup of G such that $C_S(M) \cap M = 1$. This is not possible because M is the only minimal normal subgroup of G and so $K = 1$. Therefore

$$S \cong G/M = G/O_p(G)$$

is a p'-group and so G has p-length one, a contradiction. Thus $G \in \mathcal{R}$. Therefore $\mathcal{L} \subseteq \mathcal{R}$ and thus

$$\mathcal{L} = \mathcal{R}.$$

□

Chapter 6

Supersoluble groups in $\mathcal{W}_2$

In this chapter we give a characterisation of supersoluble groups of Wielandt length two and of order coprime to six which generalises that for the nilpotent groups given by Ormerod [12]. Our classification is similar to the one for T-groups given in Theorem 2.2.4.

In the first section we prove some basic results for supersoluble groups of odd order and Wielandt length two. All these results are based on the property of supersoluble groups given in Lemma 2.3.3. In section one we prove that the nilpotent residual of a supersoluble group G of odd order and Wielandt length two is complemented in G.

In later sections we give our characterisation of supersoluble groups of Wielandt length two and order coprime to six. To do this we define the idea of a generalised matched extension of nilpotent groups. We prove that every generalised matched extension of nilpotent groups whose orders are coprime to six is a supersoluble group in $\mathcal{W}_2$, and that each supersoluble group in $\mathcal{W}_2$ whose order is coprime to six has an expression as a generalised matched extension.

6.1 Some basic results

The main result of this section is Theorem 6.1.7 which says that in a supersoluble group of odd order and Wielandt length two, the nilpotent residual is complemented. We begin with the following lemma which gives useful information about the nilpotent residual of a supersoluble group.

Lemma 6.1.1 *Let A be a normal Sylow p-subgroup of a non-nilpotent supersoluble group G and B be a Hall p'-subgroup of G so that $G = BA$.*

If N is the nilpotent residual of G and H is the nilpotent residual of B, then

$$N = H[B, A].$$

Proof By Lemma 4.1.1, we have

$$N = (N \cap B)(N \cap A).$$

We claim that

$$H = N \cap B.$$

Now

$$B/N \cap B \cong BN/N \subseteq G/N$$

and is therefore nilpotent. This means

$$H \subseteq N \cap B.$$

Again

$$G/HA = BA/HA = BHA/HA \cong B/B \cap HA$$

and by Theorem 2.1.7 we have

$$B \cap HA = H.$$

Therefore G/HA is nilpotent and so

$$N \subseteq HA.$$

Now

$$N = N \cap HA = (N \cap H)(N \cap A),$$

by Lemma 4.1.1, and so

$$N \cap B = N \cap H$$

again using Theorem 2.1.7. Hence

$$N = H(N \cap A),$$

as required.

Now as B centralises $A/[B,A]$, we have

$$G/[B,A] = (B[B,A]/[B,A]) \times (A/[B,A]).$$

It follows that

$$H[B,A]/[B,A] \lhd G/[B,A],$$

and so

$$H[B,A] \lhd G.$$

Then

$$G/(H[B,A]) = (B[B,A]/H[B,A]) \times (AH/(H[B,A])).$$

Here

$$B[B,A]/H[B,A] = BH[B,A]/H[B,A] \cong B/B \cap (H[B,A])$$

which, using Theorem 2.1.7, is equal to

$$B/(H(B \cap [B,A])) = B/H.$$

Therefore $G/H[B,A]$ is nilpotent. This means

$$N \subseteq H[B,A].$$

and so

$$N = N \cap (H[B,A]) = H(N \cap [B,A]),$$

by Theorem 2.1.7. But then

$$N \cap A = N \cap [B,A] \subseteq [B,A].$$

Since N is the nilpotent residual of G, there exists an integer n such that $\gamma_n(G) = N$. But, by Theorem 2.1.10 (2), we know that

$$[B,A] = [B,A,jB].$$

for all $j \geq 1$. Therefore

$$[B,A] \subseteq \gamma_n(G) = N$$

and hence

$$[B, A] \subseteq N \cap A.$$

Thus we have

$$N \cap A = [B, A],$$

as required. □

The next lemma is true for all finite soluble groups $G \in \mathcal{W}_2$, which can be written as a semidirect product.

Lemma 6.1.2 *Let $G = BA$ be a soluble semidirect product of subgroups A, B of coprime order, where $A \lhd G$. If $G \in \mathcal{W}_2$, then every element of B induces by conjugation on $A/\omega(A)$, a universal power automorphism.*

Proof We know from Theorem 4.1.5 that

$$\omega(G) = P\omega(A)$$

where $P = B \cap \omega(G)$ acts as a group of power automorphisms on A and $\omega(A) = \omega(G) \cap A$. Also $G/\omega(G)$ is a T-group. It follows from Theorem 2.2.4 that, for all $b \in B$, $b\omega(G)$ acts by conjugation on $A\omega(G)/\omega(G) \cong A/\omega(A)$ as a power automorphism which, by Theorem 2.2.3, is universal. Now

$$[P, A] \subseteq A \cap \omega(G) = \omega(A),$$

so

$$P = B \cap \omega(G) \subseteq C_B(A/\omega(A)).$$

Therefore every $b \in B$ acts as a universal power automorphism on $A/\omega(A)$. □

The following lemma is very useful. It tells us that if a Hall p'-subgroup of a supersoluble group G of odd order and Wielandt length two acts non-trivially on a normal Sylow p-subgroup A of G, then A has nilpotency class at most two.

Lemma 6.1.3 *Let A be a normal Sylow p-subgroup of a non-nilpotent supersoluble group G of odd order and Wielandt length two. Let B be a Hall p'-subgroup of G, so that $G = BA$. If B acts non-trivially on $A/\omega(A)$, then A has nilpotency class at most two.*

Proof Since G is a supersoluble group of odd order and B acts non-trivially on A, A can not be a 3-group. Since A is normal, $A \in \mathcal{W}_2$. Therefore $A/\omega(A)$ is abelian and, by Theorem 3.2.1, A has nilpotency class at most three.

Let us suppose, contrary to the claim of the lemma that A has nilpotency class exactly three. It follows from Theorem 3.2.4 that A has elements a_1, a_2 for which

$$[a_1, a_2, a_1] = a_1^{p^r}$$

where $r = e(A)$. Now suppose $b \in B$ induces a non-trivial automorphism by conjugation on $A/\omega(A)$. Then, by Lemma 6.1.2, there exists an integer m which is not divisible by p, for which $a_1^b = a_1^m c$ and $a_2^b = a_2^m d$ where $c, d \in \omega(A) \subseteq Z_2(A)$, by Theorem 2.1.16. Then, using the regularity of A and Lemmas 3.2.1 and 3.3.3:

$$[a_1, a_2, a_1]^{m^3} = [a_1^m c, a_2^m d, a_1^m c] = [a_1, a_2, a_3]^b = (a_1^{p^r})^b =$$

$$(a_1^m c)^{p^r} = (a_1^m)^{p^r} c^{p^r} [c, a_1^m]^{-p^r(p^r-1)/2} = (a_1^{p^r})^m = [a_1, a_2, a_1]^m.$$

It follows that $m^3 - m \equiv 0 \bmod p$. Hence $m^2 \equiv 1 \bmod p$ as $m \not\equiv 0 \bmod p$. This means that $b^2 \in C_B((A/\omega(A))/(\Phi(A/\omega(A)))$ whence $b^2 \in C_B(A/\omega(A))$ by Theorem 2.1.3. Therefore $b \in C_B(A/\omega(A))$ because $|B|$ is odd. This contradicts our choice of b. We conclude therefore, that A has nilpotency class at most two, as claimed. □

Lemma 6.1.4 *Let A be a normal Sylow p-subgroup of a non-nilpotent supersoluble group G of odd order and Wielandt length two. Let B be a complement of A in G so that $G = BA$. Then either $[B, A] \subseteq \omega(A)$ or $C_A(B) \subseteq \omega(A)$.*

Proof It follows from Lemma 6.1.2 that B acts as a group of universal power automorphism on $A/\omega(A)$. Therefore either $[b, A] \subseteq \omega(A)$ for all $b \in B$ or, for some $b \in B$, $C_{A/\omega(A)}(b) = 1$. Therefore either $[B, A] \subseteq \omega(A)$ or $C_A(B) \subseteq \omega(A)$. □

Lemma 6.1.5 *Let $H = BK$ be a semidirect product of subgroups B, K of coprime order, with K a normal p-subgroup in H. Let K have nilpotency class exactly two. If B acts as a group of universal power automorphisms on $K/\omega(K)$ and $[B, K] = K$, then*

$$C_K(B) = 1.$$

Proof By Lemma 2.1.11, K/K' has a direct decomposition

$$K/K' = A_1/K' \times \ldots \times A_s/K'$$

into B-admissible subgroups A_i/K' with the following properties for each $i = 1, \ldots, s$:

1) A_i/K' is indecomposable as a B-module;

2) $(A_i/K')/\Phi(A_i/K')$ is an irreducible B-module.

Since G is supersoluble therefore, $(A_i/K')/\Phi(A_i/K')$ is cyclic of prime order. Hence A_i/K' is cyclic by Theorem 2.1.2. Therefore $A_i/K' = \langle y_i K'\rangle$, for $1 \leq i \leq s$, is cyclic of prime power order. Let $b \in B$. Then we may write $y_i^b = y_i^{m_i^{(b)}} c_i$ for some $c_i \in K'$ and some integers $m_i^{(b)}$, for $1 \leq i \leq s$. Note that $m_i^{(b)} \not\equiv 0 \bmod p$, for $1 \leq i \leq s$. Since $K' \subseteq \Phi(K)$, it follows that

$$K = \langle y_1, y_2, \ldots, y_s\rangle,$$

by Theorem 2.1.2. Also note that for each i there is at least one $b \in B$ such that $m_i^{(b)} \not\equiv 1 \bmod p$: otherwise $[B, K] \neq K$. We aim now to show that

$$K' = \langle [y_i, y_j] : y_i \notin \omega(K), y_j \notin \omega(K)\rangle.$$

To this end suppose that, for some j, $y_j \in \omega(K)$. Then choose $b \in B$ such that $m_j^{(b)} \not\equiv 1 \bmod p$. For simplicity we write $m_i = m_i^{(b)}$, for $1 \leq i \leq s$. By Corollary 4.3 of Ormerod [12] there is an integer n such that

$$[y_i, y_j] = y_i^n$$

for $1 \leq i \leq s$. Note that p divides n, otherwise $[y_i, y_i] \neq 1$. Hence, since K has nilpotency class two,

$$[y_i, y_j]^{m_i m_j} = [y_i, y_j]^b = (y_i^n)^b = (y_i^b)^n = (y_i^{m_i} c_i)^n$$

$$= (y_i^n)^{m_i} c_i^n [c_i, y_i^{m_i}]^{n(n-1)/2} = [y_i, y_j]^{m_i} c_i^n [c_i, y_i^{m_i}]^{n(n-1)/2}$$

whence

$$[y_i, y_j]^{m_i(m_j-1)} \in \Phi(K').$$

Now $[y_i, y_j] \notin \Phi(K')$ would mean $m_i(m_j - 1) \equiv 0 \bmod p$ leading to $m_j \equiv 1 \bmod p$, a contradiction. Hence $[y_i, y_j] \in \Phi(K')$. It follows that K' is generated by the commutators $[y_i, y_j]$ where neither y_i nor y_j belongs to $\omega(K)$.

Finally we are given that each $b \in B$ induces, by conjugation, a universal power automorphism on $K/\omega(K)$. That is, in particular, for some integer m, $m \equiv m_i^{(b)} \bmod p$ if $y_i \notin \omega(K)$. It follows that, for all such pairs i, j (when $y_i, y_j \notin \omega(K)$),

$$[y_i, y_j]^b = [y_i, y_j]^{m_i m_j} = [y_i, y_j]^{m^2} mod \Phi(K').$$

Hence $[y_i, y_j]^b = [y_i, y_j] \bmod \Phi(K')$ if and only if $m^2 \equiv 1 \bmod p$ and that is if and only if $y_i^{b^2} = y_i \bmod K'$ whence if and only if $y_i^b = y_i \bmod K'$ since $(2, |b|) = 1$. This is a contradiction to our choice of b. Hence at least one $b \in B$ acts fixed point freely on $K'/\Phi(K')$ so $[K', B] = K'$. Thus $C_{K'}(B) = 1$. Finally note that $C_{K/K'}(B) = 1$ since $K/K' = [B, K/K']$ and using Theorem 2.1.10. Therefore $C_K(B) \subseteq C_K(B) \cap K' = C_{K'}(B) = 1$, as required. □

Theorem 6.1.6 *Let A be a normal Sylow p-subgroup of a non-nilpotent supersoluble group G of odd order and Wielandt length two. Let B be a complement of A in G so that $G = BA$. Then*

$$[B, A] \cap C_A(B) = 1.$$

Proof If A is abelian, the result is an immediate consequence of Theorem 2.1.10 (c). Therefore suppose that A is non-abelian. B acts as a group of universal power automorphisms on $A/\omega(A)$ by Lemma 6.1.2. By Lemma 6.1.4, either $[B, A] \subseteq \omega(A)$ or $C_A(B) \subseteq \omega(A)$. First suppose that $[B, A] \subseteq \omega(A)$. As $\omega(A)$ is abelian, we have by Theorem 2.1.10 (c) that

$$\omega(A) = [\omega(A), B] \times C_{\omega(A)}(B).$$

But $[B, A, B] = [B, A]$ and $[B, \omega(A), B] = [B, \omega(A)]$ again by Theorem 2.1.10. Now

$$[B, A] = [B, A, B] \subseteq [\omega(A), B] \subseteq [B, A],$$

so

$$[B, A] = [B, \omega(A)].$$

As we know from above that

$$[B, \omega(A)] \cap C_{\omega(A)}(B) = 1,$$

therefore

$$[B, A] \cap C_{\omega(A)}(B) = 1.$$

But $C_{\omega(A)}(B) = \omega(A) \cap C_A(B)$. Therefore $[B, A] \cap C_A(B) = 1$ (as $[B, A] \subseteq \omega(A)$), as required.

Now suppose that $C_A(B) \subseteq \omega(A)$ so that $[B, A/\omega(A)] = A/\omega(A)$ by Theorem 2.1.10. Let $K = [B, A]$. If K is abelian then again, by Theorem 2.1.10, we have $C_K(B) = 1$, and hence

$$C_A(B) \cap K = 1,$$

as required.

We get the same result when K is non-abelian because $KandB$ satisfy the hypothesis of Lemma 6.1.5: here we have relied on Lemmas 6.1.3 and 6.1.4. □

Now we use all above results to prove the following theorem.

Theorem 6.1.7 *Let N be the nilpotent residual of a non-nilpotent supersoluble group G of odd order and Wielandt length two. Then N is complemented in G.*

Proof Let A be a normal Sylow p-subgroup of G, where p is the largest prime dividing $|G|$, and B be a Hall p'-subgroup of G. Then we have $G = BA$. By Theorem 2.1.10, we can write $G = B(C_A(B)[B, A])$. By induction on the order of G, the nilpotent residual, H say, of B must be complemented in B. Let X be a complement of H so that $B = XH$. By Lemma 6.1.1, we know that $N = H[B, A] \lhd G$. Let $Y = XC_A(B)$. Then $G = NY$ and

$$N \cap Y = H[B, A] \cap XC_A(B) = (H \cap X)([B, A] \cap C_A(B)).$$

But by Lemma 6.1.6 we have $C_A(B) \cap [B, A] = 1$. Therefore $N \cap Y = 1$. Thus we conclude that N is complemented in G. □

6.2 More results

In this section we give a characterisation theorem for supersoluble groups of Wielandt length two and order coprime to 6. This construction is similar to the one given in Theorem 2.2.3 for T-groups. This characterisation is detailed in section 3. We first need some results additional to the ones we proved in section one.

Throughout this section we use that a p-group having nilpotency class less than p is regular (by Theorem 2.1.5).

The following result gives a necessary and sufficient conditions for the direct product of an abelian group and a T-group to be a T-group.

Lemma 6.2.1 *Let G_1 be a T-group and G_2 be an abelian group. Then $G = G_1 \times G_2$ is a T-group if and only if $(|\gamma_3(G_1)|, |G_2|) = 1$ and $B_1 \times G_2$ is a Dedekind group (where B_1 is a complement of $\gamma_3(G_1)$ in G_1, ensured by 13.4.4 of Robinson [14]).*

Proof It follows from Theorem 2.2.4 that $G_1 \cong G(A, B_1, \theta)$ where $A = \gamma_3(G_1)$. Let $B = B_1 \times G_2$. Define $\phi : B \longrightarrow Paut(A)$ as follows: if $\pi : B \to B_1$ is the natural projection homomorphism then $\phi = \theta \circ \pi$. Then, by Theorem 2.2.4 again, $G(A, B, \phi)$ is a T-group and, moreover,

$$G(A, B, \phi) \cong G(A, B_1, \theta) \times G_2 \cong G.$$

Conversely suppose that G is a T-group. Then by Theorem 2.2.4, we have $(|\gamma_3(G)|, |B|) = 1$ where B is a complement of $\gamma_3(G)$ in G. But $|\gamma_3(G)| = |\gamma_3(G_1)|$ and $|B| = |B_1||G_2|$ where B_1 is a complement of $\gamma_3(G_1)$ in G_1. This means that $(|\gamma_3(G_1)|, |G_2|) = 1$ and $B_1 \times G_2$ being isomorphic to B is Dedekind. □

The following lemma gives conditions for an abelian p-group G_1 (for a prime $p > 3$) acting as a group of power automorphisms on a p-group G_2 of nilpotency class at most two, to lie in the Wielandt subgroup of the semidirect product G_1G_2.

We use the standard notation $\Omega_r(G_2)$ to denote the subgroup of G_2 generated by elements of order p^r.

Lemma 6.2.2 *Let G_1 be an abelian p-group (for $p > 3$) of exponent p^r and G_2 be a p-group of nilpotency class at most two. Let $\theta : G_1 \longrightarrow Paut(G_2)$ be a homomorphism and put $G = G_1G_2$ for the semidirect product of G_2 by G_1 under θ. Then $G_1 \subseteq \omega(G)$ if and only if G_1 centralises $\Omega_r(G_2)$.*

Proof First suppose that $G_1 \subseteq \omega(G)$. As G_1 is abelian, we have

$$G' = G_2'[G_2, G_1].$$

Also since $G_1 \subseteq Paut(G_2)$, Theorem 2.2.2 gives the following:

$$[G_2, G_1] \subseteq Z(G_2)$$

and hence $G' \subseteq Z(G_2)$ (since G_2' is also contained in $Z(G_2)$). But by Theorem 2.1.16, we have

$$G_1 \subseteq \omega(G) \subseteq Z_2(G)$$

and we know from 5.1.11(3) of Robinson [14] that $Z_2(G)$ commutes with G'. Therefore G' centralises G_1 and hence $G' \subseteq Z(G)$ and so G has nilpotency class at most two.

Since G_1 is abelian of exponent p^r, it is generated by elements of order p^r. So it is sufficient to show that elements of order p^r of G_1 centralise $\Omega_r(G_2)$. Let g_1 be an element of G_1 such that $|g_1| = p^r$ and let $g_2 \in \Omega_r(G_2)$. Note that the exponent of $\Omega_r(G_2)$ divides p^r, as G_2 is regular. Now put $g = g_1g_2$. For $x \in G_1$, there exists an integer m such that $g^x = g^m$ and so $(g_1g_2)^x = g_1g_2^x = (g_1g_2)^m$. But as G is a regular p-group, G_1 acts as a group of universal power automorphisms on G by Theorem 2.2.3. So we have $g_2^x = g_2^m$. But using Theorem 2.1.14, we have

$$g_1g_2^m = (g_1g_2)^m = g_1^m g_2^m [g_2, g_1]^{-m(m-1)/2}.$$

As $G_1 \cap G_2 = 1$, we have

$$g_1^{m-1} = [g_2, g_1]^{m(m-1)/2} = 1.$$

This means that $m \equiv 1 \mod |g_1|$ and therefore $m \equiv 1 \mod |g_2|$ (as $|g_2|$ divides $|g_1|$). Thus we conclude that $g_2^x = g_2$ and so G_1 acts trivially on $\Omega_r(G_2)$.

Conversely suppose that G_1 centralises $\Omega_r(G_2)$. For any $g \in G$, there exist $g_1 \in G_1$ and $g_2 \in G_2$ such that $g = g_1g_2$. Let $x \in G_1$. By hypothesis if $g_2 \in \Omega_r(G_2)$, then $g^x = g$. Suppose that g_2 does not belong to $\Omega_r(G_2)$ and let $|g_2| = p^s$ for $s > r$. But $g_2^{p^{s-r}}$ is an element of G_2 such that $|g_2^{p^{s-r}}| = p^r$ and hence belongs to $\Omega_r(G_2)$. This means that $(g_2^{p^{s-r}})^x = g_2^{p^{s-r}}$. But as G_2 is a regular p-group, G_1 acts as a group of universal power automorphisms on G_2 by Theorem 2.2.3. Therefore there exists a positive integer m such that $g_3^x = g_3^m$ for all $g_3 \in G_2$ and so $(g_2^{p^{s-r}})^x = g_2^{mp^{s-r}} = g_2^{p^{s-r}}$. Therefore $g_2^{p^{s-r}(m-1)} = 1$ and hence $(m-1)p^{s-r} = tp^s$ for some positive integer t. This means $m - 1 = tp^r$ and so $m = 1 + tp^r$. Hence $g^x = g_1^x g_2^x = g_1 g_2^x = g_1^m g_2^m$.

We claim that $g_1^m g_2^m = (g_1g_2)^m$. By Theorem 2.1.14, we have

$$g_1^m g_2^m = (g_1g_2)^m c_2^{\binom{m}{2}} c_3^{\binom{m}{3}} y$$

where c_2 and c_3 are products of commutators with entries g_1 and g_2 of weight two and three respectively, and $y \in \gamma_4(G)$. But since $g_1^{m-1} = 1$, we immediately see from Theorem 2.1.6 that

$$c_2^{\binom{m}{2}} c_3^{\binom{m}{3}} = 1$$

as $p \geq 5$. This means that $g_1^m g_2^m = (g_1g_2)^m y$ and hence $(g\gamma_4(G))^x = g^m\gamma_4(G)$. Thus

$$G_1\gamma_4(G)/\gamma_4(G) \subseteq \omega(G/\gamma_4(G)).$$

Since $G/\gamma_4(G)$ has a factorisation satisfying the hypothesis of the theorem we have, from the first paragraph of the 'only if' part of this theorem proved above, that $G/\gamma_4(G)$ has nilpotency class two. In other words

$$\gamma_3(G) \subseteq \gamma_4(G)$$

and since G is nilpotent, we immediately see that $\gamma_3(G) = 1$ and hence G has nilpotency class at most two. This proves our claim that

$$g^x = g_1^m g_2^m = (g_1g_2)^m = g^m$$

and thus

$$G_1 \subseteq \omega(G).$$

□

We explicitly record one of the features of the proof.

Corollary 6.2.3 *Let G_1 be an abelian p-group of exponent p^r and G_2 be a p-group of nilpotency class at most two on which G_1 acts as a group of power automorphisms. If G_1 centralises $\Omega_r(G_2)$, then the semidirect product of G_2 by G_1 has nilpotency class at most two.* □

6.3 A structure theorem

We now have enough information in hand to construct all finite supersoluble groups of Wielandt length two and order coprime to six.

To begin we introduce a definition which abstracts the properties elucidated in Theorem 6.1.6 and in Lemmas 6.1.3, 3.3.4 and 6.2.2. In this section all groups will have order coprime to six.

Definition 6.3.1 *We say that a p-group A has a* special factorisation Y_0N_0 *if the following properties hold.*

1) N_0 is of nilpotency class at most two and $Y_0 \in \mathcal{W}_2$;

2) $N_0 \lhd A$, $N_0 \cap Y_0 = 1$ and $A = Y_0N_0$;

3) Conjugation by the elements of Y_0 induces power automorphisms on N_0;

4) If Y_0 is non-abelian, then

a) $A' \subseteq Y_0$;

b) and if Y_0 has nilpotency class 3 then the exponent of N_0 is at most $p^{e(Y_0)}$.

5) If Y_0 is abelian then

$$[Y_0, \Omega_r(N_0)] = 1$$

where p^r is the exponent of Y_0.

Here $\Omega_r(N_0) = \langle y : y^{p^r} = 1\rangle$.

Lemma 6.3.2 *a) Let A be a p-group having a special factorisation. Then $A \in \mathcal{W}_2$.*

b) If p is the largest prime dividing the order of a supersoluble $\mathcal{W}_2$-group G and if A_0 is a normal Sylow p-subgroup of G, and B_0 a Hall p'-subgroup of G, then $N_0 = [B_0, A_0]$ and $Y_0 = C_{A_0}(B_0)$ afford a special factorisation of A_0.

Proof a) If Y_0 is abelian, then by (5) of the definition, and Corollary 6.2.3, A has nilpotency class at most two and so $A \in \mathcal{W}_2$. Therefore suppose that Y_0 is non-abelian. Then $A' \subseteq Y_0$. This means that N_0 is abelian and $[N_0, Y_0] = 1$. It follows from Corollary 3.3.5 and (4) of the definition that $A \in \mathcal{W}_2$.

b) By Theorem 6.1.6 we know that $A_0 = Y_0 N_0$ with $N_0 \lhd A_0$ and $N_0 \cap Y_0 = 1$. Hence, by (a), $Y_0 \in \mathcal{W}_2$ since $Y_0 \cong A_0/N_0$. From Lemma 6.1.4 either $N_0 \subseteq \omega(A_0)$ or $Y_0 \subseteq \omega(A_0)$. In the first case

$$[N_0, Y_0] \subseteq N_0 \cap Y_0 = 1,$$

so $A_0 = N_0 \times Y_0$. Since also $N_0' = 1$, we have $A_0' = N_0' Y_0' [N_0, Y_0] = Y_0' \subseteq Y_0$. In particular $Y_0' \neq 1$ ensures $A_0' \subseteq Y_0$. What is more, if Y_0 has nilpotency class three then we conclude from Corollary 3.3.5 that the exponent of N_0 divides $p^{e(Y_0)}$ so that (4) holds. Clearly in the case when $Y_0' \neq 1$, (3) is satisfied too.

Finally if $Y_0 \subseteq \omega(A_0)$ then Lemma 6.2.2 ensures that (5) holds, and also that (3) is satisfied in this case. □

Definition 6.3.3 $G = B_0 A_0$ *is a* matched extension *of A_0 by B_0 if, for some prime p:*

1) A_0 is a normal p-subgroup having special factorisation $Y_0 N_0$ with

$$N_0 = [A_0, B_0]$$

and

$$Y_0 = C_{A_0}(B_0);$$

2) B_0 is a supersoluble p'-group in $\mathcal{W}_2$;

3) $B_0/C_{B_0}(N_0)$ is abelian of exponent dividing $p-1$;

4) if Y_0 is abelian then the elements of B_0 induce, by conjugation, power automorphisms in $N_0/N_0 \cap \omega(A_0)$;

5) $(|\gamma_3(B_0/\omega(B_0))|, |B_0/C_{B_0}(A_0)|) = 1$.

Lemma 6.3.4 *If $G = B_0A_0$ is a matched extension, then G is a supersoluble $\mathcal{W}_2$-group.*

Proof The aim of the proof is to calculate $\omega(G)$ and to show that $G/\omega(G)$ is a T-group. First of all we have, from Theorem 4.1.5, that

$$\omega(G) = P_0\omega(A_0)$$

where P_0 is the subgroup of $\omega(B_0)$ inducing power automorphisms in A_0, by conjugation.

Note that

$$C_{\omega(B_0)}(A_0) \subseteq P_0.$$

Also

$$C_{\omega(B_0)}(A_0) = C_{B_0}(A_0) \cap \omega(B_0).$$

Now $B_0/\omega(B_0)$ is a T-group, by hypothesis, and $B_0/C_{B_0}(A_0)$ is abelian by (3) of the definition of a matched extension. It follows from (5) of the definition of matched extension, and Lemma 6.2.1, that

$$B_0/\omega(B_0) \times B_0/C_{B_0}(A_0)$$

is a T-group. Hence, by Theorem 2.2.5, $C_{\omega(B_0)}(A_0)$ is a T-group. It then follows that B_0/P_0 is a T-group since B_0/P_0 is isomorphic to a homomorphic image of $B_0/C_{\omega(B_0)}(A_0)$.

If A_0 is abelian, then $\omega(G) = P_0A_0$ and so $G/\omega(G) \cong B_0/P_0$ and therefore $G \in \mathcal{W}_2$.

Now suppose that A_0 is non-abelian. In this case

$$P_0 = C_{\omega(B_0)}(A_0),$$

by Theorem 5.3.2 of Cooper [6]. There are two cases to consider: either $Y_0' \neq 1$ or $Y_0' = 1$.

In the first case it follows from property (4) of the definition of special factorisation, that $N_0 \subseteq \omega(A_0)$. Hence

$$G/\omega(G) \cong Y_0/Y_0 \cap \omega(A_0) \times B_0/P_0,$$

a direct product of T-groups of relatively coprime orders, so $G/\omega(G)$ is also a T-group, using Theorem 4.1.5, for example.

In the second case, when Y_0 is abelian, we have from property (5) of the definition of special factorisation, and Lemma 6.2.2, that $Y_0 \subseteq \omega(A_0)$. Hence

$$A_0\omega(G)/\omega(G) \cong A_0/\omega(A_0) \cong N_0/N_0 \cap \omega(A_0),$$

an abelian p-group. However

$$G/\omega(G) = (B_0\omega(G)/\omega(G))(A_0\omega(G)/\omega(G),$$

and $B_0\omega(G)/\omega(G) \cong B_0/P_0$ is a T-group of p'-order acting by conjugation on $A_0\omega(G)/\omega(G)$ as power automorphisms: property (4) of matched extension. Hence again by Theorem 4.1.5, $G/\omega(G)$ is a T-group.

This completes the proof that $G \in \mathcal{W}_2$. To see that G is supersoluble, use Theorem 2.3.6. □

We now generalise the concept of a matched extension. Suppose we have a group $G = YN$ where $N \lhd G$, $N \cap Y = 1$ and where N, Y are both nilpotent. Moreover suppose that $p_1, p_2, \ldots, p_r \geq 5$ are the primes in decreasing order which divide $|G|$. We write N_i, Y_i respectively for the Sylow p_i-subgroups of N, Y where $1 \leq i \leq r$. The group G will be called a *generalised matched extension* of N by Y according to the following inductive definition. When $r = 1$, $G = Y_1 \in \mathcal{W}_2$ and $N_1 = 1$ (so that G is a p-group with a special factorisation).

Suppose $r > 1$ and that $B_1 = (Y_2Y_3 \ldots Y_r)(N_2N_3 \ldots N_r)$ is a generalised matched extension. Then G is a generalised matched extension of N by Y if it is a matched extension of N_1Y_1 by B_1.

The last lemma now enables us to prove the following theorem.

Theorem 6.3.5 *Every generalised matched extension of nilpotent groups whose orders are coprime to 6 is a supersoluble group in $\mathcal{W}_2$.*

Proof We use induction on the number r of primes dividing the order of the generalised matched extension G. If $r = 1$ then, in the notation of the definition above, G is a p_1-group in $\mathcal{W}_2$, so we are done.

Suppose $r > 1$ and a generalised matched extension involving at most $r - 1$ primes is a supersoluble $\mathcal{W}_2$-group. Then if G is a generalised matched extension involving r primes we may write

$$G = B_1 A_1$$

where B_1 is a generalised matched extension involving the $r - 1$ primes

$$p_2, p_3, \ldots, p_r,$$

where A_1 is a p_1-group, and where the extension $B_1 A_1$ is matched. By induction B_1 is supersoluble p_1'-group in $\mathcal{W}_2$. Hence by Lemma 6.3.4, G is a supersoluble $\mathcal{W}_2$-group. This completes the induction. □

Conversely we show that generalised matched extensions give rise to all supersoluble $\mathcal{W}_2$-groups.

Theorem 6.3.6 *Let G be a supersoluble $\mathcal{W}_2$-group of order coprime to 6. Then G has an expression as a generalised matched extension.*

Proof We use induction on the number r of primes dividing $|G|$. If G is a p_1-group then, with $Y_1 = G$ and $N_1 = 1$ we see that G is a generalised matched extension.

Suppose $r > 1$ and that supersoluble $\mathcal{W}_2$-groups with fewer than r prime divisors of their orders are generalised matched extensions.

If p_1 is the largest prime dividing $|G|$ let A_1 be the normal Sylow p_1-subgroup of G and let B_1 be a Hall p_1'-subgroup of G, so that $G = B_1 A_1$. Write $N_1 = [B_1, A_1]$ and $Y_1 = C_{A_1}(B_1)$. By induction B_1 has an expression as a generalised matched extension.

By Lemma 6.3.2 (b) $A_1 = Y_1N_1$ is a special factorisation of A_1 and B_1 is a supersoluble p_1'-group, so (1),(2) of the definition of matched extension hold. We need to verify that the remaining axioms (3)-(5) of matched extension hold for $G = B_1A_1$.

First of all (3) holds because, by Theorem 2.3.6, N_1B_1 is supersoluble.

To prove (4) we observe first that if A_1 is abelian, then (4) is automatically satisfied as

$$N_1 \cap \omega(A_1) = N_1 \cap A_1 = N_1$$

and so $N_1/N_1 \cap \omega(A_1)$ is trivial. So suppose that Y_1 is abelian but A_1 is non-abelian. Then $Y_1 \subseteq \omega(A_1)$ is immediate from Lemma 6.1.4 if N_1 is non-abelian; and if N_1 is abelian and $N_1 \subseteq \omega(A_1)$ then $[N_1, Y_1] \subseteq N_1 \cap Y_1 = 1$, so A_1 is abelian, a contradiction. Hence, in this case, $Y_1 \subseteq \omega(A_1)$. Therefore

$$A_1\omega(G)/\omega(G) \cong A_1/\omega(G) \cap A_1 \cong A_1/\omega(A_1) \cong N_1/N_1 \cap \omega(A_1).$$

Next observe that $\omega(G) = P_1\omega(A_1)$ where P_1 is the subgroup of $\omega(B_1)$ inducing power automorphisms on A_1. Since A_1 is non-abelian, $P_1 = C_{\omega(B_1)}(A_1)$ by Theorem 5.3.2 of Cooper [6]. Hence, since $G/\omega(G)$ is a T-group and

$$G/\omega(G) = (A_1\omega(G)/\omega(G))(B_1\omega(G)/\omega(G)),$$

it follows that B_1 acts as a group of power automorphisms on $N_1/N_1 \cap \omega(A_1)$, as required to confirm (4).

Finally to prove (5) note that

$$C_{\omega(B_1)}(A_1) = C_{B_1}(A_1) \cap \omega(B_1),$$

so the T-group $B_1/P_1 = B_1/C_{\omega(B_1)}(A_1)$ is a semidirect product of $B_1/\omega(B_1)$ and $B_1/C_{B_1}(A_1)$. The latter group is abelian by Lemma 2.3.4 (b). Then (5) follows from the following lemma. □

Lemma 6.3.7 *If H is a T-group and M_1, M_2 are normal subgroups of H for which $M_1 \cap M_2 = 1$ and H/M_2 is abelian, then*

$$(|H/M_2|, |\gamma_3(H/M_1)|) = 1.$$

Proof Firstly $\gamma_3(H) \subseteq M_2$ and therefore $M_1 \cap \gamma_3(H) = 1$. Hence

$$\gamma_3(H/M_1) = \gamma_3(H)M_1/M_1 \cong \gamma_3(H)$$

whereas H/M_1 is isomorphic to a factor group of $H/\gamma_3(H)$. The result follows since

$$(|\gamma_3(H)|, |H/\gamma_3(H)|) = 1$$

by Theorem 2.2.4. □

With this Lemma we have concluded our characterisation of supersoluble $\mathcal{W}_2$-groups of order coprime to 6: Theorems 6.3.5 and 6.3.6 show that they are precisely the groups with a generalised matched extension.

Chapter 7

A sketch for further research

The aim of this short chapter is to indicate, with several examples, how the main result of this thesis, namely that contained in Theorems 6.3.5 and 6.3.6 might be extended, and the limits to this extension.

We show in Theorem 6.1.7 that a supersoluble $\mathcal{W}_2$-group of odd order splits over its nilpotent residual. The expression of such a group, at least in the case considered, namely when the order is coprime to six, as a generalised matched extension, is integrally bound up with this splitting.

The following example shows that for supersoluble $\mathcal{W}_2$-groups in general the nilpotent residual is not complemented. Therefore we can not expect a characterisation of all supersoluble $\mathcal{W}_2$-groups in quite the same form.

Example 7.1 Let X be non-abelian group of order 27 and exponent 3 and let Y be a cyclic group of order 9. Then we may present their direct product as

$$A = X \times Y = \langle a, x, y, u | a^3 = x^3 = y^3 = u^9 = 1, x^a =$$
$$x, y^a = yx, x^y = x, [a, u] = [x, u] = [y, u] = 1\rangle.$$

Now let $B = \langle b \rangle$ be cyclic of order two. We may define an action of B on $X \times Y$ as follows:

$$a^b = a^{-1}, y^b = y^{-1}, x^b = x, u^b = u.$$

This may be confirmed by using Van Dyck's Theorem.

Let $H = BA$ be the semidirect product of A by B using this action. Note that

$$N = \langle xu^3 \rangle \subseteq Z(H).$$

Then $G = H/N$ has a presentation

$$G = \langle a, b, y, u : a^3 = y^3 = u^9 = [a, u] = [y, u] = [b, u] = 1,$$

$$y^a = yu^{-3}, a^b = a^{-1}, y^b = y^{-1} \rangle.$$

Then the subgroup X_0 of G defined by $X_0 = \langle a, y \rangle$ is the nilpotent residual $G^{\mathcal{N}}$ of G. This is because $G/X_0 \cong C_2 \times C_3$ so $G^{\mathcal{N}} \subseteq X_0$; and on the other hand $X_0 = [X_0, b]$ so $X_0 \subseteq \gamma_n(G)$ for all $n \geq 1$, so $X_0 \subseteq G^{\mathcal{N}}$. However X_0 is not complemented in G. If it were there would be an element t of order 3 in G with $t \notin X_0$. However t would belong to $X_0\langle u \rangle$, the normal Sylow 3-subgroup of G, so $t = vw$ where $v \in X_0$ and $w \in \langle u \rangle$. Then $1 = t^3 = v^3 w^3 = w^3$ whence $w = u^3 \in X_0$ or $w = u^{-3} \in X_0$, so $t \in X_0$, a contradiction. That is $G^{\mathcal{N}}$ is not complemented.

Finally we observe that G is supersoluble, using Theorem 2.3.6, for example; and $G \in \mathcal{W}_2$. This follows because $\langle u \rangle \subseteq Z(G)$ and $G/\langle u \rangle$ is a T-group, using Theorem 2.2.4, for example. Hence $G/\langle u \rangle$ is a T-group. But $\langle u \rangle \subseteq \omega(G)$, so $G/\omega(G)$ is a T-group and therefore $G \in \mathcal{W}_2$.

On a more positive note we may observe that soluble groups of odd order and Wielandt length two do split in a way which might be advantageous.

Lemma 7.2 *Let A be the supersoluble residual of a finite soluble group G having Wielandt length at most two. If G has odd order, then A has a complement B in G so that $G = BA$.*

Proof Since G has Wielandt length at most two, therefore $G/\omega(G)$ is a T-group and hence supersoluble. Also by Theorem 2.2.10 $\omega(G)/F(\omega(G))$ lies in the centre of $G/F(\omega(G))$. Thus we conclude that $G/F(\omega(G)$ is a supersoluble group.

But, by the definition of A, it is the smallest normal subgroup of G such that G/A is supersoluble, therefore $A \subseteq F(\omega(G))$. Also since G is odd, A is abelian.

Now by Theorem 2.1.4, A has a complement B in G so that $G = BA$. □

This result may mean that a theorem analogous to Theorem 6.3.5 is obtainable for soluble groups in $\mathcal{W}_2$ where the order of the group in question is odd.

Another example showing that groups of even order are likely to have rather different structure is the following: it shows that for arbitrary finite soluble groups the supersoluble residual need not be complemented.

Example 7.3 Let $X = GL_2(3)$ and $Y = \langle y \rangle$ be a cyclic group of order 4. Let $a \in X$ be defined by

$$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so that $X = SL_2(3)\langle a \rangle$. Define a subgroup H of $X \times Y$ by

$$H = SL_2(3)\langle ay \rangle.$$

Now

$$b = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and y^2 are both central in H, so that $\langle by^2 \rangle$ is normal in H. Write

$$G = H/\langle by^2 \rangle,$$

so that G is a soluble group of order 48.

To calculate $\omega(G)$ we need to know something about the subnormal subgroups of G. To describe them we introduce the following notation: let

$$K = SL_2(3)\langle by^2 \rangle / \langle by^2 \rangle.$$

Then it is straight-forward to check that

$$1 \lhd K^{(2)} \lhd K' \lhd K \lhd G$$

is the unique chief series of G. Hence if S sn G and $S \neq G$, then $S \subseteq K$. However if S sn K and $S \neq K$, $S \subseteq K'$. Then note that $K \cong SL_2(3)$ since $SL_2(3) \cap \langle by^2 \rangle = 1$, so $K' \cong Q_8$, the quarternion group of order 8. Since Q_8 is a T-group it follows that K' normalises every subnormal subgroup of G, so $K' \subseteq \omega(G)$. Moreover $G/K' \cong S_3$, the symmetric group on three elements, a T-group, so $G/\omega(G)$ is also a T-group. Therefore $G \in \mathcal{W}_2$.

From the chief series above we also see that the supersoluble residual of G is K'. If K' were complemented in G then a complement would be isomorphic to S_3

and therefore G would have an element of order 2 outside K'. It is easy to check this is not so.

That is G is an example of a finite soluble group whose supersoluble residual is not complemented.

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